Journal of Pure and Applied Algebra 214 (2010) 344–369

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Standard isotrivial fibrations with $p_g = q = 1$, II

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ARTICLE INFO

Article history:
Received 13 June 2008
Received in revised form 8 April 2009
Available online 21 July 2009
Communicated by R. Vakil

MSC:
14J29
14L30

ABSTRACT

A smooth, projective surface S is called a *standard isotrivial fibration* if there exists a finite group G which acts faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. Standard isotrivial fibrations of general type with $p_g = q = 1$ have been classified in [F. Polizzi, Standard isotrivial fibrations with $p_g = q = 1$, J. Algebra 321 (2009),1600–1631] under the assumption that T has only Rational Double Points as singularities. In the present paper we extend this result, classifying all cases where S is a minimal model. As a by-product, we provide the first examples of minimal surfaces of general type with $p_g = q = 1$, $K_S^2 = 5$ and Albanese fibration of genus 3. Finally, we show with explicit examples that the case where S is not minimal actually occurs.

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0. Introduction

Surfaces of general type with $p_g = q = 1$ are still not well understood, and few examples are known. For a minimal surface S satisfying $p_g(S) = q(S) = 1$, one has $2 \leq K_S^2 \leq 9$ and the Albanese map is a connected fibration onto an elliptic curve. We denote by g_{alb} the genus of a general Albanese fibre of S. A classification of surfaces with $K_S^2 = 2, 3$ has been obtained by Catanese, Ciliberto, Pignatelli in [1–4]. For higher values of K_S^2 some families are known, see [5–11]. As the title suggest, this paper considers surfaces with $p_g = q = 1$ which are *standard isotrivial fibrations*. This means that there exists a finite group G which acts faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$, where G acts diagonally on the product (see [12]). When $p_g = q = 1$ and T contains at worst Rational Double Points (RDPs) as singularities, standard isotrivial fibrations have been studied in [9,6]. In the present article we make a further step toward their complete classification, since we describe all cases where S is a minimal model. As a by-product, we provide the first examples of minimal surfaces of general type with $p_g = q = 1$, $K_S^2 = 5$ and $g_{\mathrm{alb}} = 3$ (see Section 5.2).

Our classification procedure combines methods from both geometry and combinatorial group theory. The basic idea is that since S is the minimal desingularization of $T = (C \times F)/G$, the two projections $\pi_C \colon C \times F \longrightarrow C$, $\pi_F \colon C \times F \longrightarrow F$ induce two morphisms $\alpha \colon S \longrightarrow C/G$, $\beta \colon S \longrightarrow F/G$ whose smooth fibres are isomorphic to F and C, respectively. We have $1 = q(S) = g(C/G) + g(F/G)$, hence we may assume that $F/G \cong \mathbb{P}^1$ and $E := C/G$ is an elliptic curve. Therefore $\alpha \colon S \longrightarrow E$ is the Albanese fibration of S and $g_{\mathrm{alb}} = g(F)$. The geometry of S is encoded in the geometry of the two coverings $h \colon C \longrightarrow E$, $f \colon F \longrightarrow \mathbb{P}^1$ and the invariants of S impose strong restrictions on $g(C)$, $g(F)$ and $|G|$. Indeed we can prove that under our assumptions $g(F) = 2$ or 3, hence we may exploit the classification of finite groups acting on curves of low genus given in [13]. The problem of constructing our surfaces is then translated into the problem of finding two systems of generators of G, that we call $\mathcal{V}$ and $\mathcal{W}$, which are subject to strict conditions of combinatorial type. The existence of such systems of

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doi:10.1016/j.jpaa.2009.05.010

generators can be checked in every case either by hand-made computations or by using the computer algebra program GAP4 (see [14]).

This method of proof is similar to the one used in [6,9], of which the present paper is a natural sequel; the main problem here is that when T contains singularities worse than RDPs, they contribute not only to $\chi(\mathcal{O}_S)$, but also to K_S^2. However, since in any case T contains only cyclic quotient singularities, this contribution is well known and can be computed in terms of Hirzebruch–Jung continued fractions (Corollary 3.6). When S is minimal, we are able to use all this information in order to achieve a complete classification.

Theorem. *Let $\lambda : S \longrightarrow T := (C \times F)/G$ be a standard isotrivial fibration of general type with $p_g = q = 1$, and assume that T contains at least one singularity which is not a RDP and that S is a minimal model. Then there are exactly the following cases.*

K_S^2	$g_{\text{alb}} = g(F)$	$g(C)$	G	IdSmall Group (G)	$\text{Sing}(T)$
5	3	3	$\mathcal{S}_3$	$G(6, 1)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	5	$D_{4,3,-1}$	$G(12, 1)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	5	D_6	$G(12, 4)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	9	$D_{2,12,5}$	$G(24, 5)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	9	$\mathcal{S}_4$	$G(24, 12)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	17	$\mathbb{Z}_2 \times \mathcal{S}_4$	$G(48, 48)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	33	$\mathcal{S}_3 \ltimes (\mathbb{Z}_4)^2$	$G(96, 64)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
5	3	57	$\text{PSL}_2(\mathbb{F}_7)$	$G(168, 42)$	$\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
3	2	11	$\mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$	$G(24, 8)$	$2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
3	2	21	$\text{GL}_2(\mathbb{F}_3)$	$G(48, 29)$	$2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$
2	2	7	$D_{2,8,3}$	$G(16, 8)$	$2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$
2	2	10	$\text{SL}_2(\mathbb{F}_3)$	$G(24, 3)$	$2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$
2	2	3	$\mathcal{S}_3$	$G(6, 1)$	$2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$
2	2	5	$D_{4,3,-1}$	$G(12, 1)$	$2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$
2	2	5	D_6	$G(12, 4)$	$2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$

Examples of non-minimal standard isotrivial fibrations with $p_g = q = 1$ actually exist. We exhibit two of them, one with $K_S^2 = 2$ (see Section 5.5) and one with $K_S^2 = 1$ (see Section 6.1); in both cases $g_{\text{alb}} = 3$ and the corresponding minimal model $\widehat{S}$ satisfies $K_{\widehat{S}}^2 = 3$. The description of all non-minimal examples would put an end to the classification of standard isotrivial fibrations with $p_g = q = 1$; however, it seems to us difficult to achieve it by using our method. The main problem is that we are not able to find an effective lower bound for K_S^2. In fact, we can easily show that S contains at most five (-1)-curves (Proposition 6.1); nevertheless, when we contract them further (-1)-curves may appear. For instance, this actually happens in our example with $K_S^2 = 1$.

Notations and conventions. All varieties, morphisms, etc. in this article are defined over $\mathbb{C}$. If S is a projective, non-singular surface then K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler characteristic*. Throughout the paper we use the following notation for groups:

- $\mathbb{Z}_n$: cyclic group of order n.
- $D_{p,q,r} = \mathbb{Z}_p \ltimes \mathbb{Z}_q = \langle x, y \mid x^p = y^q = 1,\ xyx^{-1} = y^r \rangle$: split metacyclic group of order pq. The group $D_{2,n,-1}$ is the dihedral group of order $2n$ and it will be denoted by D_n.
- $\mathcal{S}_n$, $\mathcal{A}_n$: symmetric, alternating group on n symbols. We write the composition of permutations from the right to the left; for instance, $(13)(12) = (123)$.
- $\text{GL}_n(\mathbb{F}_q)$, $\text{SL}_n(\mathbb{F}_q)$, $\text{PSL}_n(\mathbb{F}_q)$: general linear, special linear and projective special linear groups of $n \times n$ matrices over a field with q elements.
- Whenever we give a presentation of a semi-direct product $H \ltimes N$, the first generators represent H and the last generators represent N. The action of H on N is specified by conjugation relations.
- The order of a finite group G is denoted by $|G|$. If $x \in G$, the order of x is denoted by $|x|$, its centralizer in G by $C_G(x)$ and the conjugacy class of x by $\text{Cl}(x)$. If $x, y \in G$, their commutator is defined as $[x, y] = xyx^{-1}y^{-1}$. The set of elements of G different from the identity is denoted by $G^{\times}$.
- If $X = \{x_1, \ldots, x_n\} \subset G$, the subgroup generated by X is denoted by $\langle x_1, \ldots, x_n \rangle$. The derived subgroup of G is denoted by $[G, G]$.
- IdSmallGroup (G) indicates the label of the group G in the GAP4 database of small groups. For instance IdSmallGroup $(D_4) = G(8, 3)$ means that D_4 is the third in the list of groups of order 8.

1. Group-theoretic preliminaries

In this section we fix the algebraic set-up and we present some group-theoretic preliminaries.

Definition 1.1. Let G be a finite group and let

$$\mathfrak{g}' \geq 0, \quad m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$$

be integers. A *generating vector* for G of type $(\mathfrak{g}' \mid m_1, \ldots, m_r)$ is a $(2\mathfrak{g}' + r)$-tuple of elements

$$\mathcal{V} = \{g_1, \ldots, g_r;\; h_1, \ldots, h_{2\mathfrak{g}'}\}$$

such that the following conditions are satisfied:

- the set $\mathcal{V}$ generates G;
- $|g_i| = m_i$;
- $g_1 g_2 \cdots g_r \Pi_{i=1}^{\mathfrak{g}'} [h_i, h_{i+\mathfrak{g}'}] = 1$.

If such a $\mathcal{V}$ exists, then G is said to be $(\mathfrak{g}' \mid m_1, \ldots, m_r)$-*generated*.

Remark 1.2. If an *abelian* group G is $(\mathfrak{g}' \mid m_1, \ldots, m_r)$-generated then either $r = 0$ or $r \geq 2$. Moreover if $r = 2$ then $m_1 = m_2$.

For convenience we make abbreviations such as $(4 \mid 2^3, 3^2)$ for $(4 \mid 2, 2, 2, 3, 3)$ when we write down the type of the generating vector $\mathcal{V}$. Moreover we set $\mathbf{m} := (m_1, \ldots, m_r)$.

Proposition 1.3 (*Riemann Existence Theorem*). *A finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $\mathfrak{g}$ if and only if there exist integers $\mathfrak{g}' \geq 0$ and $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$ such that G is $(\mathfrak{g}' \mid m_1, \ldots, m_r)$-generated, with generating vector $\mathcal{V} = \{g_1, \ldots, g_r;\; h_1, \ldots, h_{2\mathfrak{g}'}\}$, and the Riemann–Hurwitz relation holds:*

$$2\mathfrak{g} - 2 = |G| \left(2\mathfrak{g}' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{1}$$

If this is the case, $\mathfrak{g}'$ is the genus of the quotient Riemann surface $Y := X/G$ and the G-cover $X \longrightarrow Y$ is branched in r points $P_1, \ldots, P_r$ with branching numbers $m_1, \ldots, m_r$, respectively. In addition, the subgroups $\langle g_i \rangle$ and their conjugates provide all the nontrivial stabilizers of the action of G on X.

We refer the reader to [13, Section 2], [15, Chapter 3], [16] and [9, Section 1] for more details.

Now let X be a compact Riemann surface of genus $\mathfrak{g} \geq 2$ and let $G \subseteq \mathrm{Aut}(X)$. For any $h \in G$ set $H := \langle h \rangle$ and define the set of fixed points of h as

$$\mathrm{Fix}_X(h) = \mathrm{Fix}_X(H) := \{x \in X \mid hx = x\}.$$

For our purposes it is also important to take into account how an automorphism acts in a neighborhood of each of its fixed points. We follow the exposition of [15, pp.17,38]. Let $\mathcal{D}$ be the unit disk and $h \in \mathrm{Aut}(X)$ of order $m > 1$ such that $hx = x$ for a point $x \in X$. Then there is a unique primitive complex mth root of unity ξ such that any lift of h to $\mathcal{D}$ that fixes a point in $\mathcal{D}$ is conjugate to the transformation $z \longrightarrow \xi \cdot z$ in $\mathrm{Aut}(\mathcal{D})$. We write $\xi_x(h) = \xi$ and we call ξ^{-1} the *rotation constant* of h in x. Then for each integer $q \leq m - 1$ such that $(q, m) = 1$ we define

$$\mathrm{Fix}_{X,q}(h) = \{x \in \mathrm{Fix}_X(h) \mid \xi_x(h) = \xi^q\},$$

that is the set of fixed points of h with rotation constant ξ^{-q}. Clearly, we have

$$\mathrm{Fix}_X(h) = \biguplus_{\substack{q \leq m-1 \\ (q,m)=1}} \mathrm{Fix}_{X,q}(h).$$

Proposition 1.4. *Assuming that we are in the situation of Proposition 1.3, let $h \in G^{\times}$ be of order m, $H = \langle h \rangle$ and $(q, m) = 1$. Then*

$$|\mathrm{Fix}_X(h)| = |N_G(H)| \cdot \sum_{\substack{1 \leq i \leq r \\ m \mid m_i \\ H \sim_G \langle g_i^{m_i/m} \rangle}} \frac{1}{m_i}$$

and

$$|\mathrm{Fix}_{X,q}(h)| = |C_G(h)| \cdot \sum_{\substack{1 \leq i \leq r \\ m \mid m_i \\ h \sim_G g_i^{m_i q/m}}} \frac{1}{m_i}.$$

Proof. See [15, Lemmas 10.4 and 11.5]. □

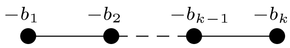
Corollary 1.5. *Assume that* $h \sim_G h^q$. *Then* $|Fix_{X,1}(h)| = |Fix_{X,q}(h)|$.

2. Cyclic quotient singularities of surfaces and Hirzebruch–Jung resolutions

Let n and q be natural numbers with $0 < q < n$, $(n, q) = 1$ and let ξ_n be a primitive nth root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \xi_n \rangle$ on $\mathbb{C}^2$ defined by $\xi_n \cdot (x, y) = (\xi_n x, \xi_n^q y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ has a cyclic quotient singularity of type $\frac{1}{n}(1, q)$, and $X_{n,q} \cong X_{n',q'}$ if and only if $n = n'$ and either $q = q'$ or $qq' \equiv 1 \pmod n$. The exceptional divisor on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is a H–J string (abbreviation of Hirzebruch–Jung string), that is to say, a connected union $E = \bigcup_{i=1}^k Z_i$ of smooth rational curves $Z_1, \dots, Z_k$ with self-intersection ≤ -2, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i, and $Z_i Z_j = 0$ if $|i - j| \geq 2$. More precisely, given the continued fraction

$$\frac{n}{q} = [b_1, \dots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\dots - \cfrac{1}{b_k}}}, \quad b_i \geq 2,$$

the dual graph of E is



(see [17, Chapter II] and [23, Chapter III]). Notice that a RDP of type A_n corresponds to the cyclic quotient singularity $\frac{1}{n+1}(1, n)$.

Definition 2.1. Let x be a cyclic quotient singularity of type $\frac{1}{n}(1, q)$. Then we set

$$h_x = 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^{k} (b_i - 2),$$
$$e_x = k + 1 - \frac{1}{n},$$
$$B_x = 2e_x - h_x = \frac{1}{n}(q + q') + \sum_{i=1}^{k} b_i,$$

where $1 \leq q' \leq n - 1$ is such that $qq' \equiv 1 \pmod n$.

3. Standard isotrivial fibrations

In this section we establish the basic properties of standard isotrivial fibrations. Definition 3.1 and Theorem 3.2 can be found in [12].

Definition 3.1. We say that a projective surface S is a *standard isotrivial fibration* if there exists a finite group G acting faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$, where G acts diagonally on the product. The two maps $\alpha: S \longrightarrow C/G$, $\beta: S \longrightarrow F/G$ will be referred as the *natural projections*.

The stabilizer $H \subseteq G$ of a point $y \in F$ is a cyclic group [18, p. 106]. If H acts freely on C, then T is smooth along the scheme-theoretic fiber of $\sigma: T \longrightarrow F/G$ over $\bar{y} \in F/G$, and this fiber consists of the curve C/H counted with multiplicity $|H|$. Thus, the smooth fibers of σ are all isomorphic to C. On the contrary, if $x \in C$ is fixed by some non-zero element of H, then T has a cyclic quotient singularity over the point $\overline{(x, y)} \in (C \times F)/G$. These observations lead to the following statement, which describes the singular fibers that can arise in a standard isotrivial fibration (see [12], Theorem 2.1).

Theorem 3.2. *Let* $\lambda: S \longrightarrow T = (C \times F)/G$ *be a standard isotrivial fibration and let us consider the natural projection* $\beta: S \longrightarrow F/G$. *Take any point over* $\bar{y} \in F/G$ *and let* Λ *denote the schematic fiber of* β *over* $\bar{y}$. *Then*

(i) *The reduced structure of* Λ *is the union of an irreducible curve* Y, *called the central component of* Λ, *and either none or at least two mutually disjoint H–J strings, each meeting* Y *at one point, and each being contracted by* λ *to a singular point of* T. *These strings are in one-to-one correspondence with the branch points of* $C \longrightarrow C/H$, *where* $H \subseteq G$ *is the stabilizer of* y.
(ii) *The intersection of a string with* Y *is transversal, and it takes place at only one of the end components of the string.*
(iii) Y *is isomorphic to* C/H, *and has multiplicity equal to* $|H|$ *in* Λ.

An analogous statement holds if we consider the natural projection $\alpha: S \longrightarrow C/G$.

Corollary 3.3. *If T has just two singularities, i.e.*

$$\mathrm{Sing}(T) = \frac{1}{n_1}(1, q_1) + \frac{1}{n_2}(1, q_2)$$

then $n_1 = n_2$.

If T has just three singularities, i.e.

$$\mathrm{Sing}(T) = \frac{1}{n_1}(1, q_1) + \frac{1}{n_2}(1, q_2) + \frac{1}{n_3}(1, q_3)$$

then, for all $i = 1, 2, 3$, the integer n_i divides l.c.m.$\{n_k \mid k \neq i\}$.

Proposition 3.4. *Let $\lambda : S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration. Assume that*

(1) *all elements of order n are conjugate in G;*
(2) *T contains a singular point of type $\frac{1}{n}(1, q)$ for some q such that $(q, n) = 1$.*

Then T contains a singular point of type $\frac{1}{n}(1, r)$ for all r such that $(r, n) = 1$.

Proof. By assumption (2) there exists a point $p = (p_1, p_2) \in C \times F$ such that the stabilizer of p has order n and its generator h acts, in suitable local coordinates centered at p, as $h \cdot (x, y) = (\xi x, \xi^q y)$, where $\xi = e^{2\pi i/n}$. Therefore $p_2 \in |\mathrm{Fix}_{F,q}(h)|$. Now let r be such that $(r, n) = 1$; using assumption (1) and Corollary 1.5 we obtain $|\mathrm{Fix}_{F,r}(h)| = |\mathrm{Fix}_{F,q}(h)| \neq 0$. If $p_2' \in \mathrm{Fix}_{F,r}(h)$, then in suitable local coordinates centered in $p' := (p_1, p_2')$ the element h acts as $h \cdot (x, y') = (\xi x, \xi^r y')$. This means that the image of p' in T is a singular point of type $\frac{1}{n}(1, r)$. $\square$

For a proof of the following result, see [19, p. 509–510] and [20]:

Proposition 3.5. *Let V be a smooth algebraic surface, and let G be a finite group acting on V with only isolated fixed points. Let $\lambda : S \longrightarrow T$ be the minimal desingularization. Then we have*

(i) $K_S^2 = \frac{1}{|G|} \cdot K_V^2 + \sum_{x \in \mathrm{Sing}\, T} h_x$.
(ii) $e(S) = \frac{1}{|G|} \cdot e(V) + \sum_{x \in \mathrm{Sing}\, T} e_x$.
(iii) $H^0(S, \Omega_S^1) = H^0(V, \Omega_V^1)^G$.

So we obtain

Corollary 3.6. *Let $\lambda : S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration. Then the invariants of S are given by*

(i) $K_S^2 = \frac{8(g(C)-1)(g(F)-1)}{|G|} + \sum_{x \in \mathrm{Sing}\, T} h_x$.
(ii) $e(S) = \frac{4(g(C)-1)(g(F)-1)}{|G|} + \sum_{x \in \mathrm{Sing}\, T} e_x$.
(iii) $q(S) = g(C/G) + g(F/G)$.

Remark 3.7. If $g(C/G) > 0$ and $g(F/G) > 0$ then S is necessarily a minimal model. If instead $g(F/G) = 0$ [respectively $g(C/G) = 0$] it may happen that the central component of some reducible fiber of α [respectively β] is a (-1)-curve. Examples of this situation are given in Sections 5.5 and 6.1.

4. The case $\chi(\mathcal{O}_S) = 1$

Proposition 4.1. *Let $\lambda : S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $\chi(\mathcal{O}_S) = 1$ and $K_S^2 \geq 2$. Then the possible singularities of T are included in the following list:*

- $K_S^2 = 6$:
  1. $2 \times \frac{1}{2}(1, 1)$.
- $K_S^2 = 5$:
  1. $\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$;
  2. $2 \times \frac{1}{4}(1, 1)$;
  3. $3 \times \frac{1}{2}(1, 1)$.
- $K_S^2 = 4$:
  1. $\frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$;
  2. $2 \times \frac{1}{5}(1, 2)$;
  3. $\frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$;
  4. $4 \times \frac{1}{2}(1, 1)$.

- $K_S^2 = 3$:
  1. $2 \times \frac{1}{4}(1, 3)$;
  2. $\frac{1}{5}(1, 1) + \frac{1}{5}(1, 4)$;
  3. $\frac{1}{7}(1, 2) + \frac{1}{7}(1, 3)$;
  4. $\frac{1}{8}(1, 1) + \frac{1}{8}(1, 3)$;
  5. $\frac{1}{8}(1, 5) + \frac{1}{8}(1, 3)$;
  6. $\frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$;
  7. $2 \times \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$;
  8. $2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$;
  9. $5 \times \frac{1}{2}(1, 1)$.
- $K_S^2 = 2$:
  1. $\frac{1}{6}(1, 1) + \frac{1}{6}(1, 5)$;
  2. $\frac{1}{9}(1, 2) + \frac{1}{9}(1, 4)$;
  3. $2 \times \frac{1}{10}(1, 3)$;
  4. $\frac{1}{11}(1, 3) + \frac{1}{11}(1, 7)$;
  5. $\frac{1}{12}(1, 5) + \frac{1}{12}(1, 7)$;
  6. $2 \times \frac{1}{13}(1, 5)$;
  7. $\frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 3)$;
  8. $\frac{1}{2}(1, 1) + \frac{1}{5}(1, 2) + \frac{1}{10}(1, 3)$;
  9. $\frac{1}{2}(1, 1) + \frac{1}{8}(1, 1) + \frac{1}{8}(1, 3)$;
  10. $\frac{1}{2}(1, 1) + \frac{1}{8}(1, 3) + \frac{1}{8}(1, 5)$;
  11. $\frac{1}{3}(1, 2) + 2 \times \frac{1}{6}(1, 1)$;
  12. $\frac{1}{4}(1, 1) + 2 \times \frac{1}{8}(1, 3)$;
  13. $3 \times \frac{1}{5}(1, 2)$;
  14. $2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$;
  15. $2 \times \frac{1}{2}(1, 1) + 2 \times \frac{1}{5}(1, 2)$;
  16. $4 \times \frac{1}{4}(1, 1)$;
  17. $\frac{1}{3}(1, 1) + \frac{1}{3}(1, 2) + 2 \times \frac{1}{4}(1, 1)$;
  18. $2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$;
  19. $3 \times \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$;
  20. $3 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$;
  21. $6 \times \frac{1}{2}(1, 1)$.

Moreover the case $K_S^2 = 8$ occurs if and only if the action of G is free, i.e. if and only if T is non-singular, whereas the case $K_S^2 = 7$ does not occur.

Proof. By Corollary 3.6 we have $K_S^2 = 2e(S) - \sum_{x \in \mathrm{Sing}\, T}(2e_x - h_x)$ and Noether formula yields $K_S^2 = 12 - e(S)$, hence

$$K_S^2 = 8 - \frac{1}{3} \sum_{x \in \mathrm{Sing}\, T} B_x, \tag{2}$$

where B_x is as in Definition 2.1.

Notice that $3 \leq B_x \leq 18$ and that $B_x = 3$ if and only if x is of type $\frac{1}{2}(1, 1)$. By Theorem 3.2 there are either none or at least 2 singularities, and if there are exactly two singularities they are of the form $\frac{1}{n}(1, q_1)$ and $\frac{1}{n}(1, q_2)$, see Corollary 3.3. By analyzing all singularities with $B_x \leq 6$, we see that one cannot have exactly two singularities x_1 and x_2 with $B_{x_1} > 12$ and $B_{x_2} < 6$. Hence we may only consider singularities with $B_x \leq 12$. A list of all such singularities with their numerical invariants is given in Appendix A.

For each fixed K_S^2 we have to consider all possibilities for $\mathrm{Sing}(T)$ such that $\sum_{x \in \mathrm{Sing}(T)} B_x = 24 - 3K_S^2$ and we must exclude those sets of singularities contradicting Corollary 3.3. In this way we get our list. If $K_S^2 = 8$ then Eq. (2) implies that T is smooth, whereas if $K_S^2 = 7$ then T would have exactly one singular point of type $\frac{1}{2}(1, 1)$, impossible by Theorem 3.2. □

Proposition 4.2. *Let S be as in Proposition 4.1 and let us assume $|\mathrm{Sing}\, T| = 2$ or 3. Then*

- *m_i divides $g(C) - 1$ for all $i \in \{1, \dots, r\}$, except at most one;*
- *n_j divides $g(F) - 1$ for all $j \in \{1, \dots, s\}$, except at most one.*

If $|\mathrm{Sing}\, T| = 4$ or 5 then the same statement holds with "at most two" instead of "at most one".

Proof. Assume $|\mathrm{Sing}\, T| = 2$ or 3. Then by Theorem 3.2 the corresponding H–J strings must belong to the same fiber of $\beta: S \longrightarrow F/G$. It follows that, for all i except one, there is a subgroup H of G, isomorphic to $\mathbb{Z}_{m_i}$, which acts freely on C. Now Riemann–Hurwitz formula applied to $C \longrightarrow C/H$ gives

$$g(C) - 1 = m_i(g(C/H) - 1),$$

so m_i divides $g(C) - 1$. The statement about the n_j is analogous. If $|\mathrm{Sing}\, T| = 4$ or 5 then the H–J strings belong to at most two different fibers of β and the same proof applies. $\square$

Corollary 4.3. *If $|\mathrm{Sing}\, T| \leq 3$ and $g(F) = 2$ then $s = 1$, that is $\mathbf{n} = (n_1)$. In particular, under these assumptions G is not abelian (see Remark 1.2).*

5. Standard isotrivial fibrations with $p_g = q = 1$

From now on we suppose that $\lambda: S \longrightarrow T = (C \times F)/G$ is a standard isotrivial fibration with $p_g = q = 1$. Since $q = 1$, we may assume that $E := C/G$ is an elliptic curve and that $F/G \cong \mathbb{P}^1$. Then the natural projection $\alpha: S \longrightarrow E$ is the Albanese morphism of S and $g_{\mathrm{alb}} = g(F)$. Let $\mathcal{V} = \{g_1, \ldots, g_r\}$ be a generating vector for G of type $(0 \mid m_1, \ldots, m_r)$, inducing the G-cover $F \longrightarrow \mathbb{P}^1$ and let $\mathcal{W} = \{\ell_1, \ldots, \ell_s;\ h_1, h_2\}$ be a generating vector of type $(1 \mid n_1, \ldots, n_s)$ inducing the G-cover $C \longrightarrow E$. Then Riemann–Hurwitz formula implies

$$\begin{aligned} 2g(F) - 2 &= |G|\left(-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right) \\ 2g(C) - 2 &= |G| \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \end{aligned} \tag{3}$$

The cyclic subgroups $\langle g_1 \rangle, \ldots, \langle g_r \rangle$ and their conjugates provide the non-trivial stabilizers of the action of G on F, whereas $\langle \ell_1 \rangle, \ldots, \langle \ell_s \rangle$ and their conjugates provide the non-trivial stabilizers of the actions of G on C. The singularities of T arise from the points in $C \times F$ with nontrivial stabilizer; since the action of G on $C \times F$ is the diagonal one, it follows that the set $\mathscr{S}$ of all nontrivial stabilizers for the action of G on $C \times F$ is given by

$$\mathscr{S} = \left(\bigcup_{\sigma \in G} \bigcup_{i=1}^{r} \langle \sigma g_i \sigma^{-1} \rangle\right) \cap \left(\bigcup_{\sigma \in G} \bigcup_{j=1}^{s} \langle \sigma \ell_j \sigma^{-1} \rangle\right) \cap G^{\times}. \tag{4}$$

Proposition 5.1. *Let G be a finite group which is both $(0 \mid m_1, \ldots, m_r)$-generated and $(1 \mid n_1, \ldots, n_s)$-generated, with generating vectors $\mathcal{V} = \{g_1, \ldots, g_r\}$ and $\mathcal{W} = \{\ell_1, \ldots, \ell_s;\ h_1, h_2\}$, respectively. Denote by*

$$f: F \longrightarrow \mathbb{P}^1 = F/G,$$
$$h: C \longrightarrow E = C/G$$

the G-covers induced by $\mathcal{V}$ and $\mathcal{W}$ and let $g(F)$, $g(C)$ be the genera of F and C, that are related to $|G|$, $\mathbf{m}$, $\mathbf{n}$ by (3). Define

$$k = \frac{8(g(C) - 1)(g(F) - 1)}{|G|} + \sum_{x \in \mathrm{Sing}(T)} h_x$$

and assume that equality

$$k = 8 - \frac{1}{3} \sum_{x \in \mathrm{Sing}(T)} B_x \tag{5}$$

holds. Then the minimal desingularization S of T satisfies

$$p_g(S) = q(S) = 1, \qquad K_S^2 = k.$$

Moreover, if $k > 0$ then S is of general type.

Proof. The normal surface T satisfies $q(T) = 1$; since all quotient singularities are rational it follows $q(S) = 1$. Corollary 3.6 and relation (5) yield $K_S^2 = k$ and $K_S^2 + e(S) = 12$, hence $\chi(\mathcal{O}_S) = 1$ by Noether formula; this implies $p_g(S) = 1$. Finally if $k > 0$ then S is of general type, because $q(S) > 0$. $\square$

Lemma 5.2. *Let $\lambda: S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$. Then we have*

$$K_S^2 - \sum_{x \in \mathrm{Sing}(T)} h_x = 4(g(F) - 1) \sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \tag{6}$$

Proof. Applying Corollary 3.6 and the second relation in (3) we obtain

$$\begin{aligned} K_S^2 - \sum_{x \in \mathrm{Sing}(T)} h_x &= 4(g(F)-1) \cdot 2\frac{(g(C)-1)}{|G|} \\ &= 4(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right). \quad \square \end{aligned}$$

The cases where T has at worst RDP have already been classified in [9,6]. Hence, in the sequel we will consider the situation where T contains at least one singularity which is not a RDP.

Proposition 5.3. *Let $\lambda\colon S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, $K_S^2 \geq 2$ such that T contains at least one singularity which is not a RDP. Then there are at most the following possibilities:*

- $K_S^2 = 5$

$$g(F)=3, \qquad \mathbf{n}=(3), \qquad \mathrm{Sing}(T) = \frac{1}{3}(1,1)+\frac{1}{3}(1,2);$$

$$g(F)=3, \qquad \mathbf{n}=(8), \qquad \mathrm{Sing}(T) = 2\times\frac{1}{4}(1,1).$$

- $K_S^2 = 4$

$$g(F)=3, \qquad \mathbf{n}=(4), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1)+2\times\frac{1}{4}(1,1).$$

- $K_S^2 = 3$

$$g(F)=2, \qquad \mathbf{n}=(2,4), \qquad \mathrm{Sing}(T) = 2\times\frac{1}{2}(1,1)+2\times\frac{1}{4}(1,1);$$

$$g(F)=2, \qquad \mathbf{n}=(6), \qquad \mathrm{Sing}(T) = 2\times\frac{1}{2}(1,1)+\frac{1}{3}(1,1)+\frac{1}{3}(1,2).$$

- $K_S^2 = 2$

$$g(F)=3, \qquad \mathbf{n}=(16), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1)+\frac{1}{8}(1,1)+\frac{1}{8}(1,3);$$

$$g(F)=2, \qquad \mathbf{n}=(8), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1)+\frac{1}{8}(1,3)+\frac{1}{8}(1,5);$$

$$g(F)=3, \qquad \mathbf{n}=(12), \qquad \mathrm{Sing}(T) = \frac{1}{3}(1,2)+2\times\frac{1}{6}(1,1);$$

$$g(F)=2, \qquad \mathbf{n}=(4), \qquad \mathrm{Sing}(T) = 2\times\frac{1}{2}(1,1)+\frac{1}{4}(1,1)+\frac{1}{4}(1,3);$$

$$g(F)=3, \qquad \mathbf{n}=(4), \qquad \mathrm{Sing}(T) = 4\times\frac{1}{4}(1,1);$$

$$g(F)=2, \qquad \mathbf{n}=(4^2), \qquad \mathrm{Sing}(T) = 4\times\frac{1}{4}(1,1);$$

$$g(F)=2, \qquad \mathbf{n}=(3), \qquad \mathrm{Sing}(T) = 2\times\frac{1}{3}(1,1)+2\times\frac{1}{3}(1,2).$$

Proof. For every value of K_S^2 we must analyze all possible singularities of T as listed in Proposition 4.1. Moreover we have to exclude the cases in which all singularities of T are RDPs, namely $K_S^2 = 6$, $K_S^2 = 5$ (iii), $K_S^2 = 4$ (iv), $K_S^2 = 3$ (ix) and $K_S^2 = 2$ (xxi), where T contains only singular points of type A_1, and $K_S^2 = 3$ (i), where T contains only singular points of type A_3.

- $K_S^2 = 5$

(i) $\mathrm{Sing}(T) = \frac{1}{3}(1,1)+\frac{1}{3}(1,2)$. Using formula (6) and the table in Appendix A we obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{4}{3}.$$

If $s = 1$ then $\frac{4}{3} < g(F)-1 \leq \frac{8}{3}$, which implies $g(F) = 3$, $\mathbf{n} = (3)$. If $s \geq 2$ then $g(F)-1 \leq \frac{4}{3}$, so $g(F) = 2$ which contradicts Corollary 4.3.

(ii) $\mathrm{Sing}(T) = 2 \times \frac{1}{4}(1, 1)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{7}{4}.$$

If $s = 1$ then $\frac{7}{4} < g(F) - 1 \le \frac{7}{2}$, hence $g(F) = 3$ or 4. The case $g(F) = 4$ is numerically impossible, so the only possibility is $g(F) = 3$, $\mathbf{n} = (8)$. If $s \ge 2$ then $g(F) - 1 \le \frac{7}{4}$, so $g(F) = 2$ which contradicts Corollary 4.3.

- $K_S^2 = 4$

(i) $\mathrm{Sing}(T) = \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{5}{4}.$$

If $s = 1$ then $\frac{5}{4} < g(F) - 1 \le \frac{5}{2}$, so $g(F) = 3$ which is impossible. If $s \ge 2$ then $g(F) - 1 \le \frac{5}{4}$, so $g(F) = 2$ which contradicts Corollary 4.3.

(ii) $\mathrm{Sing}(T) = 2 \times \frac{1}{5}(1, 2)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{6}{5}.$$

If $s = 1$ then $\frac{6}{5} < g(F) - 1 \le \frac{12}{5}$, so $g(F) = 3$ which is impossible. If $s \ge 2$ then $g(F) - 1 \le \frac{6}{5}$, so $g(F) = 2$ which contradicts Corollary 4.3.

(iii) $\mathrm{Sing}(T) = \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{3}{2}.$$

If $s = 1$ then $\frac{3}{2} < g(F) - 1 \le 3$, so $g(F) = 3$ or 4. In the former case we obtain the possibility $g(F) = 3$, $\mathbf{n} = (4)$; in the latter $\mathbf{n} = (2)$ and T would contain only singular points of of type A_1, a contradiction. If $s \ge 2$ then $g(F) - 1 \le \frac{3}{2}$, so $g(F) = 2$ against Corollary 4.3.

- $K_S^2 = 3$

(ii) $\mathrm{Sing}(T) = \frac{1}{5}(1, 1) + \frac{1}{5}(1, 4)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{6}{5}.$$

If $s = 1$ then $\frac{6}{5} < g(F) - 1 \le \frac{12}{5}$, so $g(F) = 3$ which is numerically impossible. If $s \ge 2$ then $g(F) - 1 \le \frac{6}{5}$, so $g(F) = 2$ that contradicts Corollary 4.3.

(iii) $\mathrm{Sing}(T) = \frac{1}{7}(1, 2) + \frac{1}{7}(1, 3)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{8}{7},$$

which gives $g(F) - 1 \le \frac{16}{7}$, so either $g(F) = 2$ or $g(F) = 3$. In the former case we must have $s = 1$, which is impossible. In the latter we obtain $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{4}{7}$, which has no integer solutions.

(iv) $\mathrm{Sing}(T) = \frac{1}{8}(1, 1) + \frac{1}{8}(1, 3)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{17}{8},$$

which implies $g(F) - 1 \le \frac{17}{4}$, hence $2 \le g(F) \le 5$. If $g(F) = 2$ then $s = 1$ by Corollary 4.3, and this is numerically impossible. It follows $g(F) = 3, 4$ or 5, hence $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{17}{16}, \frac{17}{24}$ or $\frac{17}{32}$, respectively. In all cases there are no solutions.

(v) $\mathrm{Sing}(T) = \frac{1}{8}(1, 5) + \frac{1}{8}(1, 3)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{9}{8},$$

which implies either $g(F) = 2$ or $g(F) = 3$. In the former case we have $s = 1$, which is numerically impossible. In the latter we obtain $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{9}{16}$, which has no solutions.

(vi) $\mathrm{Sing}(T) = \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = 1,$$

hence either $g(F) = 2$ or $g(F) = 3$. The former case yields $\mathbf{n} = (2^2)$ and the latter $\mathbf{n} = (2)$; then T would have at worst A_1-singularities, a contradiction.

(vii) $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{5}{4},$$

hence either $g(F) = 2$ or $g(F) = 3$. In the former case the only possibility is $\mathbf{n} = (2,4)$. In the latter we obtain $\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{5}{8}$, which has no solutions.

(viii) $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1,1) + \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{5}{6},$$

which gives the only possibility $g(F) = 2$, $\mathbf{n} = (6)$.

- $K^2 = 2$.

(i) $\mathrm{Sing}(T) = \frac{1}{6}(1,1) + \frac{1}{6}(1,5)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{7}{6},$$

hence $g(F) = 3$ or $g(F) = 2$. If $g(F) = 3$ then $\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{7}{12}$, which is a contradiction. If $g(F) = 2$ then $s = 1$ by Corollary 4.3, so $1 - \frac{1}{n_1} = \frac{7}{6}$ which is impossible.

(ii) $\mathrm{Sing}(T) = \frac{1}{9}(1,2) + \frac{1}{9}(1,4)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{10}{9},$$

hence $g(F) = 2$ or 3 and we obtain a contradiction as before.

(iii) $\mathrm{Sing}(T) = 2 \times \frac{1}{10}(1,3)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{11}{10},$$

hence $g(F) = 2$ or 3 and we obtain a contradiction as before.

(iv) $\mathrm{Sing}(T) = \frac{1}{11}(1,3) + \frac{1}{11}(1,7)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{12}{11},$$

hence $g(F) = 2$ or 3 and we obtain a contradiction as before.

(v) $\mathrm{Sing}(T) = \frac{1}{12}(1,5) + \frac{1}{12}(1,7)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{13}{12},$$

hence $g(F) = 2$ or 3 and we obtain a contradiction as before.

(vi) $\mathrm{Sing}(T) = 2 \times \frac{1}{13}(1,5)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{14}{13},$$

hence $g(F) = 2$ or 3 and we obtain a contradiction as before.

(viii) $\mathrm{Sing}(T) = \frac{1}{2}(1,1) + \frac{1}{5}(1,2) + \frac{1}{10}(1,3)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right) = \frac{9}{10},$$

hence $g(F)=2$ and $\mathbf{n}=(10)$. This means that G is one of the groups listed in Table 2 of Appendix B and that 10 divides $|G|$, a contradiction.

(ix) $\mathrm{Sing}(T)=\frac{1}{2}(1,1)+\frac{1}{8}(1,1)+\frac{1}{8}(1,3)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{15}{8},$$

so $g(F)=2,\ 3$ or 4. If $g(F)=2$ then $s=1$ (Corollary 4.3), which gives a contradiction. The case $g(F)=4$ is numerically impossible. Finally, if $g(F)=3$ we obtain $\mathbf{n}=(16)$.

(x) $\mathrm{Sing}(T)=\frac{1}{2}(1,1)+\frac{1}{8}(1,3)+\frac{1}{8}(1,5)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{7}{8}$$

and the only possibility is $g(F)=2$, $\mathbf{n}=(8)$.

(xi) $\mathrm{Sing}(T)=\frac{1}{3}(1,2)+2\times\frac{1}{6}(1,1)$. We have

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{11}{6},$$

hence $g(F)=2,\ 3$ or 4. If $g(F)=2$ then $s=1$, a contradiction. The case $g(F)=4$ is numerically impossible. Finally, if $g(F)=3$ we obtain $\mathbf{n}=(12)$.

(xii) $\mathrm{Sing}(T)=\frac{1}{4}(1,1)+2\times\frac{1}{8}(1,3)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{5}{4},$$

hence $g(F)=3$ or 2. If $g(F)=3$ then $\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{5}{8}$, which has no solutions; if $g(F)=2$ then $s=1$ which is a contradiction.

(xiii) $\mathrm{Sing}(T)=3\times\frac{1}{5}(1,2)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{4}{5},$$

hence $\mathbf{n}=(5)$ and $g(F)=2$. This means that 5 divides $|G|$ and that G is one of the groups listed in Table 2 of Appendix B, a contradiction.

(xiv) $\mathrm{Sing}(T)=2\times\frac{1}{2}(1,1)+\frac{1}{4}(1,1)+\frac{1}{4}(1,3)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{3}{4},$$

which gives the possibility $g(F)=2$, $\mathbf{n}=(4)$.

(xv) $\mathrm{Sing}(T)=2\times\frac{1}{2}(1,1)+2\times\frac{1}{5}(1,2)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{7}{10},$$

which is impossible.

(xvi) $\mathrm{Sing}(T)=4\times\frac{1}{4}(1,1)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{3}{2},$$

so $g(F)=2,\ 3$ or 4. At least one of the n_i must be divisible by 4, otherwise T could not contain singularities of type $\frac{1}{4}(1,1)$. Hence the only possibilities are $g(F)=2$, $\mathbf{n}=(4^2)$ and $g(F)=3$, $\mathbf{n}=(4)$.

(xvii) $\mathrm{Sing}(T)=\frac{1}{3}(1,1)+\frac{1}{3}(1,2)+2\times\frac{1}{4}(1,1)$. We obtain

$$(g(F)-1)\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{13}{12},$$

so either $g(F)=2$ or $g(F)=3$. Consequently, either $\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{13}{12}$ or $\sum_{j=1}^{s}\left(1-\frac{1}{n_j}\right)=\frac{13}{24}$, and in both cases there are no solutions.

(xviii) $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{2}{3},$$

which gives the possibility $g(F) = 2$, $\mathbf{n} = (3)$.

(xix) $\mathrm{Sing}(T) = 3 \times \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = 1,$$

hence $g(F) = 2$ or 3. If $g(F) = 2$ then $\mathbf{n} = (2^2)$, whereas if $g(F) = 3$ then $\mathbf{n} = (2)$; both cases are impossible otherwise T would have only A_1-singularities.

(xx) $\mathrm{Sing}(T) = 3 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$. We obtain

$$(g(F) - 1) \sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{7}{12},$$

which has no solutions.

This concludes the proof of Proposition 5.3. □

5.1. *The case where G is abelian*

Proposition 5.4. *Let $\lambda: S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, $K_S^2 \geq 2$ and G abelian. Then T contains at worst RDPs.*

Proof. Suppose that G is abelian and T contains at least one singularity which is not a RDP. Then by Proposition 5.3 and Remark 1.2 we must have

$$K_S^2 = 2, \quad g(F) = 2, \quad \mathbf{n} = (4^2), \quad \mathrm{Sing}(T) = 4 \times \frac{1}{4}(1, 1).$$

Corollary 3.6 implies $g(C) - 1 = \frac{3}{4}|G|$. Referring to Table 1 of Appendix B, we are left with two cases:

- (1c) $G = \mathbb{Z}_4$, $\quad \mathbf{m} = (2^2, 4^2)$, $\quad g(C) = 4$.
- (1h) $G = \mathbb{Z}_8$, $\quad \mathbf{m} = (2, 8^2)$, $\quad g(C) = 7$.

Therefore G must be cyclic. Let $\mathcal{W} = \{\ell_1, \ell_2;\ h_1, h_2\}$ be a generating vector of type $(1\,|\,4^2)$ for G; then $\ell_1 = (\ell_2)^{-1}$ and Proposition 1.4 implies

$$|\mathrm{Fix}_{C,1}(\ell_1)| = |\mathrm{Fix}_{C,3}(\ell_1)| = 2.$$

In particular $\mathrm{Fix}_{C,1}(\ell_1)$ and $\mathrm{Fix}_{C,3}(\ell_1)$ are both nonempty. Hence the same argument used in proof of Proposition 3.4 shows that if S contains a singularity of type $\frac{1}{4}(1, 1)$ then it contains also a singularity of type $\frac{1}{4}(1, 3)$, a contradiction. This concludes the proof. □

Therefore in the sequel we may assume that G is a nonabelian group.

5.2. *The case $K_S^2 = 5$*

Lemma 5.5. *Referring to Table 3 in Appendix B, in cases (3g), (3h), (3i), (3j), (3s), (3u), (3v) the group G is not $(1 \mid 8)$-generated.*

Proof. In cases (3g), (3h), (3i), (3j) we have $[G, G] = \mathbb{Z}_2$, in case (3s) we have $[G, G] = \mathcal{A}_4$, in case (3u) we have $[G, G] = \mathbb{Z}_4 \times \mathbb{Z}_4$ and in case (3v) we have $[G, G] = G(48, 3)$. So in all cases $[G, G]$ contains no elements of order 8 and we are done. □

Proposition 5.6. *Let $\lambda: S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, $K_S^2 = 5$ such that T contains at least one singularity which is not a RDP. Then*

$$g(F) = 3, \qquad \mathbf{n} = (3), \qquad \mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2).$$

Furthermore exactly the following cases occur:

G	IdSmall Group (G)	$\mathbf{m}$	$g(C)$	Is S minimal?
$\mathcal{S}_3$	$G(6, 1)$	$(2^4, 3)$	3	Yes
$D_{4,3,-1}$	$G(12, 1)$	$(4^2, 6)$	5	Yes
D_6	$G(12, 4)$	$(2^3, 6)$	5	Yes
$D_{2,12,5}$	$G(24, 5)$	$(2, 4, 12)$	9	Yes
$\mathcal{S}_4$	$G(24, 12)$	$(3, 4^2)$	9	Yes
$\mathcal{S}_4$	$G(24, 12)$	$(2^3, 3)$	9	Yes
$\mathbb{Z}_2 \times \mathcal{S}_4$	$G(48, 48)$	$(2, 4, 6)$	17	Yes
$\mathcal{S}_3 \ltimes (\mathbb{Z}_4)^2$	$G(96, 64)$	$(2, 3, 8)$	33	Yes
$PSL_2(\mathbb{F}_7)$	$G(168, 42)$	$(2, 3, 7)$	57	Yes

Proof. If $K_S^2 = 5$, by Proposition 5.3 we have two possibilities:

$$\text{(a)} \quad g(F) = 3, \qquad \mathbf{n} = (3), \qquad \mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2);$$

$$\text{(b)} \quad g(F) = 3, \qquad \mathbf{n} = (8), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{4}(1, 1).$$

In particular G must be one of the groups in Table 3 of Appendix B.

Let us rule out first case (b). If it occurs then $(g(C) - 1) = \frac{7}{16}|G|$ by Corollary 3.6, so $|G|$ is divisible by 16; moreover, since $\mathbf{n} = (8)$, the group G must be $(1 \mid 8)$-generated. Cases (3g), (3h), (3i), (3j), (3s), (3u), (3v) are excluded by Lemma 5.5; cases (3q), (3r), (3t) are excluded by Proposition 4.2. So (b) does not occur and we must only consider possibility (a).

If it occurs then $g(C) - 1 = \frac{1}{3}|G|$, so $|G|$ is divisible by 3. Moreover, since $\mathbf{n} = (3)$, the group G must be $(1 \mid 3)$-generated. Cases (3f), (3k), (3m), (3n), (3t), (3u) in Table 3 are excluded by Proposition 4.2. Now let us show that all the remaining cases occur.

- Case (3a). $G = \mathcal{S}_3$, $\mathbf{m} = (2^4, 3)$, $g(C) = 3$, $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$. Set

$$g_1 = (12), \qquad g_2 = (12), \qquad g_3 = (12), \qquad g_4 = (13), \qquad g_5 = (123)$$
$$\ell_1 = (123), \qquad h_1 = (13), \qquad h_2 = (12).$$

We have $\mathscr{S} = \mathrm{Cl}((123)) = \{(123),\ (132)\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

So $C \times F$ contains exactly four points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|(123)| = 2$. Hence T contains precisely two singular points and looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$, as required. So this case occurs by Proposition 5.1.

- Case (3d). $G = D_{4,3,-1} = \langle x, y \mid x^4 = y^3 = 1, xyx^{-1} = y^{-1}\rangle$, $\mathbf{m} = (4^2, 6)$, $g(C) = 5$, $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$. Set

$$g_1 = x, \qquad g_2 = xy, \qquad g_3 = y^2x^2$$
$$\ell_1 = y, \qquad h_1 = y, \qquad h_2 = x.$$

We have $\mathscr{S} = \mathrm{Cl}(y) = \{y, y^2\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 2.$$

So $C \times F$ contains exactly 8 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y| = 4$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$, as required.

- Case (3e). $G = D_6 = \langle x, y \mid x^2 = y^6 = 1, xyx^{-1} = y^{-1}\rangle$, $\mathbf{m} = (2^3, 6)$, $g(C) = 5$, $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$. Set

$$g_1 = x, \qquad g_2 = xy^2, \qquad g_3 = y^3, \qquad g_4 = y$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathscr{S} = \mathrm{Cl}(y^2) = \{y^2, y^4\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 2.$$

So $C \times F$ contains exactly 8 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y^2| = 4$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$, as required.

• Case (3l). $G = D_{2,12,5} = \langle x, y \mid x^2 = y^{12} = 1,\ xyx^{-1} = y^5\rangle$, $\mathbf{m} = (2, 4, 12)$, $g(C) = 9$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1)+\frac{1}{3}(1,2)$. Set

$$g_1 = x, \qquad g_2 = xy^{11}, \qquad g_3 = y$$
$$\ell_1 = y^4, \qquad h_1 = y, \qquad h_2 = x.$$

We have $\mathscr{S} = \mathrm{Cl}(y^4) = \{y^4, y^8\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 4.$$

So $C \times F$ contains exactly 16 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y^4| = 8$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

• Case (3o). $G = \mathcal{S}_4$, $\mathbf{m} = (3, 4^2)$, $g(C) = 9$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$. Set

$$g_1 = (123), \qquad g_2 = (1234), \qquad g_3 = (1243)$$
$$\ell_1 = (123), \qquad h_1 = (142), \qquad h_2 = (23).$$

We have $\mathscr{S} = \mathrm{Cl}((123))$, hence $|\mathscr{S}| = 8$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

So $C \times F$ contains exactly 16 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|(123)| = 8$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

• Case (3p). $G = \mathcal{S}_4$, $\mathbf{m} = (2^3, 3)$, $g(C) = 9$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$. Set

$$g_1 = (12), \qquad g_2 = (24), \qquad g_3 = (13)(24), \qquad g_4 = (123)$$
$$\ell_1 = (123), \qquad h_1 = (142), \qquad h_2 = (23).$$

We have $\mathscr{S} = \mathrm{Cl}((123))$, hence $|\mathscr{S}| = 8$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

So $C \times F$ contains exactly 16 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|(123)| = 8$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

• Case (3s). $G = \mathbb{Z}_2 \times \mathcal{S}_4$, $\mathbf{m} = (2, 4, 6)$, $g(C) = 17$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$.
Put $\mathbb{Z}_2 = \langle z \mid z^2 = 1\rangle$ and set

$$g_1 = z(14), \qquad g_2 = (1234), \qquad g_3 = z(132)$$
$$\ell_1 = (123), \qquad h_1 = z(142), \qquad h_2 = z(23).$$

We have $\mathscr{S} = \mathrm{Cl}((123))$, hence $|\mathscr{S}| = 8$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 2.$$

So $C \times F$ contains exactly 32 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|(123)| = 16$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

• Case (3v). $G = \mathcal{S}_3 \ltimes (\mathbb{Z}_4)^2 = G(96, 64)$, $\mathbf{m} = (2, 3, 8)$, $g(C) = 33$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$. Set

$$g_1 = zxz^3, \qquad g_2 = y, \qquad g_3 = xyxzxz^3$$
$$\ell_1 = y, \qquad h_1 = yz, \qquad h_2 = xy.$$

We have $\mathscr{S} = \bigcup_{\sigma \in G}\langle \sigma y \sigma^{-1}\rangle \cap G^{\times} = \mathrm{Cl}(y)$, hence $|\mathscr{S}| = 32$. In fact, G contains precisely 16 subgroups of order 3, which are all conjugate. For all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

So $C \times F$ contains exactly 64 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y| = 32$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

• Case (3w). $G = \mathrm{PSL}_2(\mathbb{F}_7)$, $\mathbf{m} = (2, 3, 7)$, $g(C) = 57$, $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$.

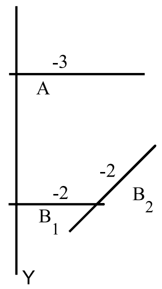


Fig. 1. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 5$.

It is well known that G can be embedded in S_8; in fact $G = \langle (375)(486),\ (126)(348)\rangle$. Set

$$g_1 = (12)(34)(58)(67), \qquad g_2 = (154)(367), \qquad g_3 = (1247358)$$
$$\ell_1 = (154)(367), \qquad h_1 = (2465837), \qquad h_2 = (1352678).$$

We have $\mathscr{S} = \mathrm{Cl}((154)(367))$, so $|\mathscr{S}| = 56$. In fact, G contains precisely 28 subgroups of order 3, which are all conjugate. For all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

So $C \times F$ contains exactly 112 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|(154)(367)| = 56$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = \frac{1}{3}(1,1) + \frac{1}{3}(1,2)$, as required.

In all cases S contains only one singular Albanese fibre $\bar{F}$, which is illustrated in Fig. 1.

Here A is a (-3)-curve, whereas B_1 and B_2 are (-2)-curves. Since $\mathbf{n} = (3)$, the central component Y has multiplicity 3 in $\bar{F}$ (see Theorem 3.2) and a straightforward computation, using $\bar{F}A = \bar{F}B_1 = \bar{F}B_2 = 0$, shows that

$$\bar{F} = 3Y + A + 2B_1 + B_2.$$

Using $K_S\bar{F} = 2g(F) - 2 = 4$ and $\bar{F}^2 = 0$ we obtain $K_SY = 1$ and $Y^2 = -1$. Hence Y is not a (-1)-curve and S is minimal. $\square$

5.3. *The case $K_S^2 = 4$*

Lemma 5.7. *Referring to Table 3 of Appendix B, in cases (3i), (3j), (3s), (3v) the group G is not $(1\,|\,4)$-generated.*

Proof. In cases (3i) and (3j) the commutator subgroup $[G, G]$ has order 2; in case (3s) we have $[G, G] = \mathcal{A}_4$, which contains no elements of order 4. In case (3v) we have $G = G(96, 64)$; if $h_1, h_2 \in G$ and $|[h_1, h_2]| = 4$ then $|\langle h_1, h_2\rangle| \leq 48$, so G is not $(1\,|\,4)$-generated. $\square$

Proposition 5.8. *Let $\lambda : S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$. If $K_S^2 = 4$ then T has only RDPs.*

Proof. Assume that $K_S^2 = 4$ and T contains at least one singularity which is not a RDP. Then by Proposition 5.3 the only possibility is

$$g(F) = 3, \qquad \mathbf{n} = (4), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1).$$

In particular G must be one of the groups in Table 3 of Appendix B. Using Corollary 3.6, we obtain $g(C) - 1 = \frac{3}{8}|G|$, so 8 divides $|G|$; moreover, since $\mathbf{n} = (4)$, it follows that G must be $(1\,|\,4)$-generated. Cases (3i), (3j), (3s), (3v) are excluded by Lemma 5.7; cases (3b), (3c), (3g), (3h), (3l), (3m), (3o), (3p) are excluded by Proposition 4.2; cases (3n) and (3w) are excluded because the signature $\mathbf{m}$ is not compatible with the singularities of T. It remains to rule out cases (3q), (3r), (3t), (3u).

• Case (3q). $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = G(32, 9)$, $\ \mathbf{m} = (2, 4, 8)$, $\ \mathrm{Sing}(T) = \frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1)$.

Let $\mathcal{W} = \{\ell_1;\ h_1,\ h_2\}$ be a generating vector of type $(1\,|\,4)$ for G. Since $[G,\ G] = \langle yz^2\rangle$, we may assume $\ell_1 = yz^2$. Then $\ell_1 \sim_G \ell_1^{-1}$, hence the same argument used in proof of Proposition 3.4 shows that if T contains a singular point of type $\frac{1}{4}(1,1)$ then it must also contain a singular point of type $\frac{1}{4}(1,3)$. Therefore this case cannot occur.

• Case (3r). $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32, 11)$, $\ \mathbf{m} = (2, 4, 8)$, $\ \mathrm{Sing}(T) = \frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1)$.

Let $\mathcal{W} = \{\ell_1;\ h_1,\ h_2\}$. Since we have $[G,\ G] = \langle yz^2\rangle$, we may assume $\ell_1 = yz^2$. Then $\ell_1 \sim_G \ell_1^{-1}$, and this case can be excluded as the previous one.

• Case (3t). $G = G(48, 33)$, $\ \mathbf{m} = (2, 3, 12)$, $\ \mathrm{Sing}(T) = \frac{1}{2}(1,1) + 2 \times \frac{1}{4}(1,1)$.

We have $[G, G] = Q_8$ and all the elements of order 4 in $[G, G]$ are conjugate in G; hence the same argument used in proof of Proposition 3.4 shows that if T contains a singular point of type $\frac{1}{4}(1, 1)$ then it must also contain a singular point of type $\frac{1}{4}(1, 3)$. Therefore this case cannot occur.

• Case (3u). $G = \mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2 = G(48, 3)$, $\mathbf{m} = (3^2, 4)$, $\mathrm{Sing}(T) = \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1)$.

Let $\mathcal{V} = \{g_1, g_2, g_3\}$ and $\mathcal{W} = \{\ell_1; h_1, h_2\}$. We have $[G, G] = \langle y, z\rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ and the conjugacy classes in G of elements of order 4 in $[G, G]$ are as follows:

$$\mathrm{Cl}(y) = \{y, z, y^3z^3\}, \qquad \mathrm{Cl}(y^3) = \{y^3, z^3, yz\}$$
$$\mathrm{Cl}(y^2z) = \{y^2z, y^3z, y^3z^2\}, \qquad \mathrm{Cl}(yz^2) = \{yz^2, yz^3, y^2z^3\}.$$

If $\ell_1 \sim_G g_3$ then T contains only singularities of type $\frac{1}{4}(1, 1)$, whereas if $\ell_1 \sim_G g_3^{-1}$ then T contains only singularities of type $\frac{1}{4}(1, 3)$. Otherwise T contains only singularities of type $\frac{1}{2}(1, 1)$. Therefore this case cannot occur. □

5.4. The case $K_S^2 = 3$

Proposition 5.9. *Let $\lambda: S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, $K_S^2 = 3$ such that T contains at least one singularity which is not a RDP. Then*

$$g(F) = 2, \qquad \mathbf{n} = (6), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2).$$

Furthermore exactly the following cases occur:

G	`IdSmall Group` (G)	$\mathbf{m}$	$g(C)$	Is S minimal?
$\mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3)$	$G(24, 8)$	$(2, 4, 6)$	11	Yes
$GL_2(\mathbb{F}_3)$	$G(48, 29)$	$(2, 3, 8)$	21	Yes

Proof. If $K_S^2 = 3$ then by Proposition 5.3 there are two possibilities, namely

$$\text{(a)} \quad g(F) = 2, \quad \mathbf{n} = (2, 4), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + 2 \times \frac{1}{4}(1, 1);$$

$$\text{(b)} \quad g(F) = 2, \quad \mathbf{n} = (6), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \times \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2).$$

In particular G must be one of the groups in Table 2 of Appendix B. We refer to this table and we consider separately the two cases.

Case (a). Using Corollary 3.6 we obtain $g(C) - 1 = \frac{5}{8}|G|$, so 8 divides $|G|$; moreover, since $\mathbf{n} = (2, 4)$, it follows that $|G|$ must be $(1 \mid 2, 4)$-generated. Cases (2b), (2c), (2g) are excluded by Proposition 4.2, whereas case (2i) is excluded because $\mathrm{GL}_2(\mathbb{F}_3)$ is not $(1 \mid 2, 4)$-generated (this can be easily checked with `GAP4`). In cases (2f) and (2h) each element of order 4 in G is conjugate to its inverse, hence the same argument used in proof of Proposition 3.4 shows that if T contains a singular point of type $\frac{1}{4}(1, 1)$ then it must also contain a singular point of type $\frac{1}{4}(1, 3)$. Therefore this case cannot occur.

Case (b). Using Corollary 3.6 we obtain $g(C) - 1 = \frac{5}{12}|G|$, so 12 divides $|G|$; moreover, since $\mathbf{n} = (6)$, it follows that G must be $(1 \mid 6)$-generated. Cases (2d), (2e), (2h) are excluded by Proposition 4.2; it remains to show that cases (2g) and (2i) actually occur.

• Case (2g). $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) = G(24, 8)$, $\mathbf{m} = (2, 4, 6)$, $g(C) = 11$, $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$. Set

$$g_1 = x, \qquad g_2 = zwx, \qquad g_3 = yzw$$
$$\ell_1 = yw, \qquad h_1 = zw, \qquad h_2 = x.$$

We have

$$\langle \ell_1 \rangle = \{1, yw, w^2, y, w, yw^2\}$$
$$\langle g_2 \rangle = \{1, zwx, y, yzwx\}$$
$$\langle g_3 \rangle = \{1, yzw, w^2, yz, w, yzw^2\}.$$

One easily checks that

- the subgroup $\langle \ell_1 \rangle$ is conjugate only to itself;
- the subgroup $\langle g_3 \rangle$ is conjugate to

$$\langle zw^2 \rangle = \{1, zw^2, w, z, w^2, zw\};$$

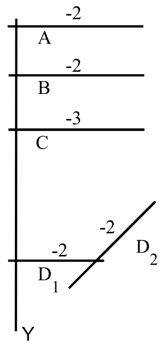


Fig. 2. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 3$.

- there are six subgroups of G conjugate to $\langle g_2 \rangle$ and different from it; all of them contain $Z(G) = \langle y \rangle$ as their unique subgroup of order 2.

Therefore $\mathscr{S} = \mathrm{Cl}(y) \cup \mathrm{Cl}(w) = \{y,\ w,\ w^2\}$. Moreover

$$|\mathrm{Fix}_F(y)| = 6$$
$$|\mathrm{Fix}_C(y)| = 4$$
$$|\mathrm{Fix}_{F,1}(w)| = |\mathrm{Fix}_{F,2}(w)| = 2$$
$$|\mathrm{Fix}_{C,1}(w)| = |\mathrm{Fix}_{C,2}(w)| = 2.$$

Hence $C \times F$ contains exactly

- 24 points having stabilizer of order $|y| = 2$ and G-orbit of cardinality 12;
- 16 points having stabilizer of order $|w| = 3$ and G-orbit of cardinality 8.

Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$, as required, so this case occurs.

• Case (2i). $G = \mathrm{GL}_2(\mathbb{F}_3)$, $\mathbf{m} = (2, 3, 8)$, $g(C) = 21$, $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$.
Set

$$g_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad g_3 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$
$$\ell_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad h_1 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

and $\ell = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. We have $(\ell_1)^3 = (g_3)^4 = \ell$ and $(\ell_1)^2 = g_2$. Therefore $\mathscr{S} = \mathrm{Cl}(\ell) \cup \mathrm{Cl}(g_2) \cup \mathrm{Cl}((g_2)^2) = \{\ell\} \cup \mathrm{Cl}(g_2)$. All the eight elements of order 3 in G are conjugate, so for all $h \in \mathrm{Cl}(g_2)$ we have

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 2$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

Moreover

$$|\mathrm{Fix}_F(\ell)| = 6, \qquad |\mathrm{Fix}_C(\ell)| = 8.$$

Therefore $C \times F$ contains exactly

- 32 points having a stabilizer of order $|g_2| = 3$ and G-orbit of cardinality 16;
- 48 points having a stabilizer of order $|\ell| = 2$ and G-orbit of cardinality 24.

Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{3}(1, 1) + \frac{1}{3}(1, 2)$, as required.

In all cases S contains only one singular Albanese fiber $\bar{F}$, which is illustrated in Fig. 2. Here A, B, D_1 and D_2 are (-2)-curves, C is a (-3)-curve and a straightforward computation shows that

$$\bar{F} = 6Y + 3A + 3B + 2C + 4D_1 + 2D_2.$$

Using $K_S\bar{F} = 2$ and $\bar{F}^2 = 0$ we obtain $K_SY = 0$ and $Y^2 = -2$, so Y is not a (-1)-curve and S is minimal. □

5.5. The case $K_S^2 = 2$

Proposition 5.10. *Let $\lambda: S \longrightarrow T = (C \times F)/G$ be a standard isotrivial fibration with $p_g = q = 1$, $K_S^2 = 2$ such that T contains at least one singularity which is not a RDP. Then there are three possibilities:*

$$(d) \quad g(F) = 2, \qquad \mathbf{n} = (4), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3)$$

$$(e) \quad g(F) = 3, \qquad \mathbf{n} = (4), \qquad \mathrm{Sing}(T) = 4 \times \frac{1}{4}(1,1)$$

$$(g) \quad g(F) = 2, \qquad \mathbf{n} = (3), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{3}(1,2).$$

In case (d) *exactly the following two subcases occur:*

G	`IdSmall Group` (G)	$\mathbf{m}$	$g(C)$	Is S minimal?
$D_{2,8,3}$	$G(16,8)$	$(2,4,8)$	7	Yes
$SL_2(\mathbb{F}_3)$	$G(24,3)$	$(3^2,4)$	10	Yes

In case (e) *there is just one occurrence:*

G	`IdSmall Group` (G)	$\mathbf{m}$	$g(C)$	Is S minimal?
$\mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2$	$G(48,3)$	$(3^2,4)$	19	No

Finally, in case (g) *there are exactly the subcases below:*

G	`IdSmall Group` (G)	$\mathbf{m}$	$g(C)$	Is S minimal?
$\mathcal{S}_3$	$G(6,1)$	$(2^2,3^2)$	3	Yes
$D_{4,3,-1}$	$G(12,1)$	$(3,4^2)$	5	Yes
D_6	$G(12,4)$	$(2^3,3)$	5	Yes

Moreover in case (e) *the minimal model $\widehat{S}$ of S satisfies $K_{\widehat{S}}^2 = 3$.*

Proof. If $K_S^2 = 2$ then by Proposition 5.3 there are seven possibilities, namely

$$\text{(a)} \quad g(F) = 3, \qquad \mathbf{n} = (16), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1) + \frac{1}{8}(1,1) + \frac{1}{8}(1,3);$$

$$\text{(b)} \quad g(F) = 2, \qquad \mathbf{n} = (8), \qquad \mathrm{Sing}(T) = \frac{1}{2}(1,1) + \frac{1}{8}(1,3) + \frac{1}{8}(1,5);$$

$$\text{(c)} \quad g(F) = 3, \qquad \mathbf{n} = (12), \qquad \mathrm{Sing}(T) = \frac{1}{3}(1,2) + 2 \times \frac{1}{6}(1,1);$$

$$\text{(d)} \quad g(F) = 2, \qquad \mathbf{n} = (4), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{2}(1,1) + \frac{1}{4}(1,1) + \frac{1}{4}(1,3);$$

$$\text{(e)} \quad g(F) = 3, \qquad \mathbf{n} = (4), \qquad \mathrm{Sing}(T) = 4 \times \frac{1}{4}(1,1);$$

$$\text{(f)} \quad g(F) = 2, \qquad \mathbf{n} = (4^2), \qquad \mathrm{Sing}(T) = 4 \times \frac{1}{4}(1,1);$$

$$\text{(g)} \quad g(F) = 2, \qquad \mathbf{n} = (3), \qquad \mathrm{Sing}(T) = 2 \times \frac{1}{3}(1,1) + 2 \times \frac{1}{3}(1,2).$$

If $g(F) = 2$ then G must be one of the groups in Table 2 of Appendix B, whereas if $g(F) = 3$ then G must be one of the groups in Table 3. Let us consider separately the different cases.

Case (a). Using Corollary 3.6 we obtain $g(C) - 1 = \frac{15}{32}|G|$, hence 32 divides $|G|$; looking at Table 3 we see that the only possibilities are (3q) and (3r). In both cases $[G, G]$ has order 4, so G is not $(1 \mid 16)$-generated and this contradicts $\mathbf{n} = (16)$. Hence this case does not occur.

Case (b). We obtain $g(C) - 1 = \frac{7}{16}|G|$, hence 16 divides $|G|$; looking at Table 2 we see that the only possibilities are (2f) and (2i). In both cases one easily checks that $[G, G]$ contains no elements of order 8, so G is not $(1 \mid 8)$-generated and this contradicts $\mathbf{n} = (8)$. Hence this case does not occur.

Case (c). We obtain $g(C) - 1 = \frac{11}{24}|G|$, so 24 divides $|G|$. Referring to Table 3 of Appendix B, we are left with cases (3l), (3m), (3n), (3o), (3p), (3s), (3t), (3u), (3v), (3w). All these possibilities can be ruled out by using Proposition 4.2, hence this case does not occur.

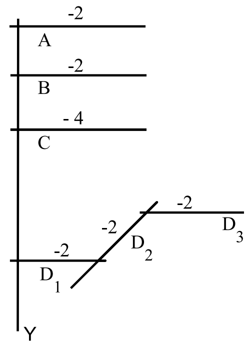


Fig. 3. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 2$, (d).

Case (d). We obtain $g(C) - 1 = \frac{3}{8}|G|$, so 8 divides $|G|$. By direct computation or using GAP4 one checks that the groups in cases (2b), (2c), (2g) and (2i) are not $(1 \mid 4)$-generated, contradicting $\mathbf{n} = (4)$; so the only possibilities are (2f) and (2h). Let us show that both actually occur.

• Case (2f). $G = D_{2,8,3}$, $\mathbf{m} = (2, 4, 8)$, $g(C) = 7$, $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$.
Set

$$g_1 = x, \qquad g_2 = xy^7, \qquad g_3 = y$$
$$\ell_1 = y^2, \qquad h_1 = y, \qquad h_2 = x.$$

We have

$$\mathrm{Cl}(y) = \{y,\ y^3\}, \qquad \mathrm{Cl}(y^2) = \{y^2,\ y^6\}, \qquad \mathrm{Cl}(y^4) = \{y^4\}.$$

Since $(g_2)^2 = (\ell_1)^2$ and $(g_3)^2 = \ell_1$, we obtain $\mathscr{S} = \bigcup_{\sigma \in G} \langle \sigma y^2 \sigma^{-1} \rangle \ \cap G^\times = \{y^2,\ y^4,\ y^6\}$. Moreover

$$|\mathrm{Fix}_F(y^4)| = 6$$
$$|\mathrm{Fix}_C(y^4)| = 4$$
$$|\mathrm{Fix}_{F,1}(y^2)| = |\mathrm{Fix}_{F,3}(y^2)| = 1$$
$$|\mathrm{Fix}_{C,1}(y^2)| = |\mathrm{Fix}_{C,3}(y^2)| = 2.$$

Therefore $C \times F$ contains exactly

- 16 points having stabilizer of order $|y^4| = 2$ and G-orbit of cardinality 8;
- 8 points having stabilizer of order $|y^2| = 4$ and G-orbit of cardinality 4.

Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$, as required.

• Case (2h). $G = \mathrm{SL}_2(\mathbb{F}_3)$, $\mathbf{m} = (3^2, 4)$, $g(C) = 10$, $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$.
Set

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \qquad g_3 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\ell_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad h_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad h_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $\ell = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The group G contains six elements of order 4, which are all conjugate. Therefore there are three cyclic subgroups $H_1,\ H_2,\ H_3$ of order 4, all conjugate and such that $H_i \cap H_j = \langle \ell \rangle$ for $i \neq j$. If $h \in G$ and $|h| = 4$ then

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,3}(h)| = 1$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,3}(h)| = 1.$$

Therefore $C \times F$ contains exactly

- 24 points having stabilizer of order $|\ell| = 2$ and G-orbit of cardinality 12;
- 12 points having stabilizer of order $|h| = 4$ and G-orbit of cardinality 6.

Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{2}(1, 1) + \frac{1}{4}(1, 1) + \frac{1}{4}(1, 3)$, as required.

Now we show that all surfaces in **Case (d)** are minimal. In fact they contain only one singular Albanese fiber $\bar{F}$, which is illustrated in Fig. 3. Here A, B, D_1, D_2 and D_3 are (-2)-curves, C is a (-4)-curve and a straightforward computation shows that

$$\bar{F} = 4Y + 2A + 2B + C + 3D_1 + 2D_2 + D_3.$$

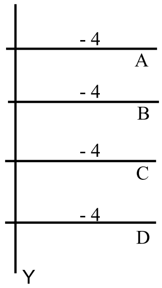


Fig. 4. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 2$, (e).

Using $K_S\bar{F} = 2$ and $\bar{F}^2 = 0$ we obtain $K_SY = 0$ and $Y^2 = -2$, so Y is not a (-1)-curve and S is minimal.

Case (e). We obtain $g(C) - 1 = \frac{3}{8}|G|$, hence 8 divides $|G|$. Referring to Table 3 in Appendix B, we have what follows.

- Cases (3b), (3c), (3g), (3h), (3i), (3j), (3l), (3m), (3s), (3v) must be excluded because the corresponding G are not $(1 \,|\, 4)$-generated, contradicting $\mathbf{n} = (4)$.

- Cases (3n) and (3w) must be excluded because no component of $\mathbf{m}$ is divided by 4, a contradiction because the singularities of T must be of type $\frac{1}{4}(1, 1)$.

- In cases (3o), (3p), (3q), (3r), (3t) all elements of order 4 in $[G, G]$ are conjugate in G; therefore the same argument used in proof of Proposition 3.4 shows that S must contain both $\frac{1}{4}(1, 1)$ and $\frac{1}{4}(1, 3)$ singularities, a contradiction.

Now we show that Case (3u) occurs.

• Case (3u). $G = \mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2 = G(48, 3)$, $\mathbf{m} = (3^2, 4)$, $g(C) = 19$, $\mathrm{Sing}(T) = 4 \times \frac{1}{4}(1, 1)$.

Set

$$\begin{aligned} &g_1 = x, \quad g_2 = x^2y^3, \quad g_3 = y \\ &\ell_1 = y, \quad h_1 = x, \quad h_2 = xyxy^2. \end{aligned}$$

We have $\mathscr{S} = \bigcup_{\sigma \in G} \langle \sigma y \sigma^{-1} \rangle \cap G^{\times}$ and the elements of order 4 in $\mathscr{S}$ are precisely $\{y, z, y^3z^3, y^3, z^3, yz\}$. Moreover $\mathrm{Cl}(y) = \{y, z, y^3z^3\}$ and $\mathrm{Cl}(y^3) = \{y^3, z^3, yz\}$. Take any $h \in \mathscr{S}$ such that $|h| = 4$; since h is not conjugate to h^{-1} in G, Proposition 1.4 implies

$$\begin{aligned} &|\mathrm{Fix}_{F,1}(h)| = 4, \quad |\mathrm{Fix}_{F,3}(h)| = 0, \\ &|\mathrm{Fix}_{C,1}(h)| = 4, \quad |\mathrm{Fix}_{C,3}(h)| = 0. \end{aligned}$$

Therefore $C \times F$ contains exactly 48 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y| = 12$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = 4 \times \frac{1}{4}(1, 1)$, as required. The surface S contain only one singular Albanese fiber $\bar{F}$, which is illustrated in Fig. 4. Here A, B, C, D are (-4)-curves and a straightforward computation shows that

$$\bar{F} = 4Y + A + B + C + D.$$

Since $K_S\bar{F} = 4$ and $(\bar{F})^2 = 0$ we obtain $K_SY = Y^2 = -1$, i.e. Y is the unique (-1)-curve in S. The minimal model $\widehat{S}$ of S is obtained by contracting Y, hence $K_{\widehat{S}}^2 = 3$. Therefore $\widehat{S}$ is an example of a minimal surface of general type with $p_g = q = 1, K^2 = g_{\mathrm{alb}} = 3$ and a unique singular Albanese fiber. The existence of such surfaces was previously established, in a completely different way, by Ishida in [21].

Case (f). We obtain $g(C) - 1 = \frac{3}{4}|G|$, hence 4 divides $|G|$; moreover G must be $(1 \,|\, 4^2)$-generated. Look at Table 2 of Appendix B. Cases (2b) and (2e) are excluded by using Proposition 4.2, whereas Case (2i) is excluded because $\mathrm{GL}_2(\mathbb{F}_3)$ is not $(1 \,|\, 4^2)$-generated. In cases (2c), (2f), (2g) and (2h) all the elements of order 4 in G are conjugate to their inverse, hence if S contains a singular point of type $\frac{1}{4}(1, 1)$ it should also contain a singular point of type $\frac{1}{4}(1, 3)$, a contradiction. Hence we must only consider (2d). In this case $G = D_{4,3,-1}$, which contains two conjugacy classes of elements of order 4, namely $\mathrm{Cl}(x) = \{x, xy, xy^2\}$ and $\mathrm{Cl}(x^3) = \{x^3, x^3y, x^3y^2\}$. Since the only element of order 2 in G is x^2, two different 2-Sylow of G intersect exactly in $\langle x^2 \rangle$. This show that T should contain some singular points of type $\frac{1}{2}(1, 1)$, a contradiction.

Case (g). We obtain $g(C) - 1 = \frac{1}{3}|G|$, hence 3 divides $|G|$; moreover G must be $(1|3)$-generated. Referring to Table 2 of Appendix B, the groups in cases (2g), (2h), (2i) are excluded because they are not $(1|3)$-generated, so we are left to show that cases (2a), (2c) and (2e) occur.

• Case (2a). $G = \mathcal{S}_3$, $\mathbf{m} = (2^2, 3^2)$, $g(C) = 3$, $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$.

Set

$$\begin{aligned} &g_1 = (12), \quad g_2 = (12), \quad g_3 = (123), \quad g_4 = (132) \\ &\ell_1 = (123), \quad h_1 = (13), \quad h_2 = (12). \end{aligned}$$

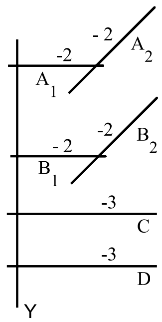


Fig. 5. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 2$, (g).

We have $\mathscr{S} = \mathrm{Cl}((123)) = \{(123), (132)\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 2$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 1.$$

Hence $C \times F$ contains exactly 8 points with nontrivial stabilizer and the G-orbit of each of these points has cardinality $|G|/|(123)| = 2$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$, as required.

• Case (2d). $G = D_{4,3,-1}$, $\mathbf{m} = (3, 4^2)$, $g(C) = 5$, $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$.
Set

$$g_1 = y, \qquad g_2 = y^2x^3, \qquad g_3 = x$$
$$\ell_1 = y, \qquad h_1 = y, \qquad h_2 = x.$$

We have $\mathscr{S} = \mathrm{Cl}(y) = \{y, y^2\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 2$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 2.$$

Hence $C \times F$ contains exactly 16 points with nontrivial stabilizer and the G-orbit of each of these points has cardinality $|G|/|y| = 4$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$, as required.

• Case (2e). $G = D_6$, $\mathbf{m} = (2^3, 3)$, $g(C) = 5$, $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$.
Set

$$g_1 = x, \qquad g_2 = xy, \qquad g_3 = y^3, \qquad g_4 = y^2$$
$$\ell_1 = y^2, \qquad h_1 = x, \qquad h_2 = y.$$

We have $\mathscr{S} = \mathrm{Cl}(y^2) = \{y^2, y^4\}$ and for all $h \in \mathscr{S}$

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = 2,$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = 2.$$

Hence $C \times F$ contains exactly 16 points with nontrivial stabilizer and the G-orbit of each of these points has cardinality $|G|/|y^2| = 4$. Looking at the rotation constants we see that $\mathrm{Sing}(T) = 2 \times \frac{1}{3}(1, 1) + 2 \times \frac{1}{3}(1, 2)$, as required.

Now we show that the surfaces in **Case (g)** are minimal. In fact they all contain only one singular Albanese fiber $\bar{F}$, which is illustrated in Fig. 5. Here A_1, A_2, B_1, B_2 are (-2)-curves, C, D are (-3)-curves and a straightforward computation shows that

$$\bar{F} = 3Y + 2A_1 + A_2 + 2B_1 + B_2 + C + D.$$

Using $K_S\bar{F} = 2$ and $\bar{F}^2 = 0$ we obtain $K_SY = 0$ and $Y^2 = -2$, so Y is not a (-1)-curve and S is minimal. □

6. The case where S is not minimal

The description of all non-minimal examples would put an end to the classification of standard isotrivial fibrations with $p_g = q = 1$; however, it seems to us difficult to achieve it by using our methods. We can prove the following

Proposition 6.1. *Let $\lambda\colon S \longrightarrow (C \times F)/G$ be a standard isotrivial fibration of general type with $p_g = q = 1$. Then S contains at most five (-1)-curves.*

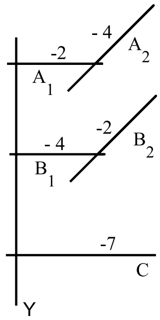


Fig. 6. The singular Albanese fiber $\bar{F}$ in the case $K_S^2 = 1$.

Proof. Let $\alpha \colon S \longrightarrow E$ be the Albanese map of S, let $\widehat{S}$ be the minimal model of S and $\hat{\alpha} \colon \widehat{S} \longrightarrow E$ the Albanese map of $\widehat{S}$. By Theorem 3.2 the (-1)-curves of S may only appear as central components of reducible fibers of α. Therefore the number of such curves is smaller than or equal to the number of singular fibers of $\hat{\alpha}$. On the other hand, by the Zeuthen–Segre formula ([22, p.116]) we have

$$10 \geq e(\widehat{S}) = \sum_{x \in \mathrm{Crit}(\hat{\alpha})} \mu_x,$$

where $\mathrm{Crit}(\hat{\alpha})$ is the set of points of E where the fiber of $\hat{\alpha}$ is singular. The integer μ_x satisfies $\mu_x \geq 1$ and equality holds if and only if the fiber of $\hat{\alpha}$ over x has an ordinary double point as a unique singularity. This would imply that the general fiber of α is rational, a contradiction. Therefore $\mu_x \geq 2$ for every $x \in \mathrm{Crit}(\hat{\alpha})$, so $\hat{\alpha}$ has at most five singular fibers. $\square$

The main problem is that further (-1)-curves may appear after contracting the (-1)-curves of S. This happens for instance in the following example.

6.1. An example with $K_S^2 = 1$

In this section we construct a standard isotrivial fibration S with $p_g = q = 1$ and $K_S^2 = 1$, whose minimal model $\widehat{S}$ satisfies $K_{\widehat{S}}^2 = 3$. The building data for S are

$$g(F) = 3, \qquad \mathbf{m} = (3^2, 7),$$
$$g(C) = 10, \qquad \mathbf{n} = (7),$$
$$G = D_{3,7,2} = \langle x, y \,|\, x^3 = y^7 = 1,\ xyx^{-1} = y^2 \rangle.$$

Set

$$g_1 = x^2, \qquad g_2 = xy^6, \qquad g_3 = y$$
$$\ell_1 = y, \qquad h_1 = y, \qquad h_2 = x.$$

We have $\mathscr{S} = \bigcup_{\sigma \in G} \langle \sigma y \sigma^{-1} \rangle \cap G^{\times} = \{y, y^2,\ y^3, y^4, y^5, y^6\}$ and moreover $\mathrm{Cl}(y) = \{y, y^2, y^4\}$, $\mathrm{Cl}(y^3) = \{y^3, y^6, y^5\}$. Hence for all $h \in \mathscr{S}$ we obtain

$$|\mathrm{Fix}_{F,1}(h)| = |\mathrm{Fix}_{F,2}(h)| = |\mathrm{Fix}_{F,4}(h)| = 1$$
$$|\mathrm{Fix}_{F,3}(h)| = |\mathrm{Fix}_{F,5}(h)| = |\mathrm{Fix}_{F,6}(h)| = 0$$
$$|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,2}(h)| = |\mathrm{Fix}_{C,4}(h)| = 1$$
$$|\mathrm{Fix}_{C,3}(h)| = |\mathrm{Fix}_{C,5}(h)| = |\mathrm{Fix}_{C,6}(h)| = 0.$$

It follows that $C \times F$ contains exactly 9 points with nontrivial stabilizer and for each of them the G-orbit has cardinality $|G|/|y| = 3$. Looking at the rotation constants we see that

$$\mathrm{Sing}(T) = \frac{1}{7}(1,1) + \frac{1}{7}(1,2) + \frac{1}{7}(1,4),$$

so using Proposition 5.1 one checks that S is a surface of general type with $p_g = q = 1$, $K_S^2 = 1$. Furthermore the surface S contains only one singular Albanese fiber $\bar{F}$ which is illustrated in Fig. 6.

Notice that, since $2 \cdot 4 \equiv 1 \pmod 7$, the cyclic quotient singularities $\frac{1}{7}(1,2)$ and $\frac{1}{7}(1,4)$ are analytically isomorphic (see Section 2); moreover, the resolution algorithm given in [17, Chapter II] implies that the corresponding Hirzebruch–Jung strings are attached in a mirror-like way to the central component Y of $\bar{F}$. A straightforward computation shows that

$$\bar{F} = 7Y + 4A_1 + A_2 + 2B_1 + B_2 + C.$$

Using $K_S\bar{F} = 4$ and $\bar{F}^2 = 0$ we obtain $K_S Y = Y^2 = -1$, hence Y is the unique (-1)-curve in S. The minimal model $\widehat{S}$ of S is obtained by first contracting Y and then the image of A_1; therefore $K_{\widehat{S}}^2 = 3$.

Acknowledgements

The authors are indebted to I. Bauer, F. Catanese and R. Pignatelli for many interesting discussions and helpful suggestions. This research started at the University of Bayreuth in September 2007, while the first author had a post-doc position and the second author was a visiting professor. Both authors were supported by the *DFG Forschergruppe "Klassifikation algebraischer Flächen und kompakter komplexer Mannigfaltigkeiten"*.

Appendix A

List of all cyclic quotient singularities $x = \frac{1}{n}(1, q)$ with $3 \leq B_x \leq 12$.

$\frac{1}{n}(1,q)$	$n/q = [b_1, \ldots, b_s]$	$\frac{1}{n}(1,q')$	$B_{\frac{1}{n}(1,q)}$	$h_{\frac{1}{n}(1,q)}$
$\frac{1}{2}(1,1)$	$[2]$	$\frac{1}{2}(1,1)$	$3+0$	0
$\frac{1}{3}(1,1)$	$[3]$	$\frac{1}{3}(1,1)$	$3+2/3$	$-1/3$
$\frac{1}{3}(1,2)$	$[2,2]$	$\frac{1}{3}(1,2)$	$5+1/3$	0
$\frac{1}{4}(1,1)$	$[4]$	$\frac{1}{4}(1,1)$	$4+1/2$	-1
$\frac{1}{4}(1,3)$	$[2,2,2]$	$\frac{1}{4}(1,3)$	$7+1/2$	0
$\frac{1}{5}(1,1)$	$[5]$	$\frac{1}{5}(1,1)$	$5+2/5$	$-9/5$
$\frac{1}{5}(1,2)$	$[3,2]$	$\frac{1}{5}(1,3)$	$6+0$	$-2/5$
$\frac{1}{5}(1,4)$	$[2,2,2,2]$	$\frac{1}{5}(1,4)$	$9+3/5$	0
$\frac{1}{6}(1,1)$	$[6]$	$\frac{1}{6}(1,1)$	$6+1/3$	$-8/3$
$\frac{1}{6}(1,5)$	$[2,2,2,2,2]$	$\frac{1}{6}(1,5)$	$11+2/3$	0
$\frac{1}{7}(1,1)$	$[7]$	$\frac{1}{7}(1,1)$	$7+2/7$	$-25/7$
$\frac{1}{7}(1,2)$	$[4,2]$	$\frac{1}{7}(1,4)$	$6+6/7$	$-8/7$
$\frac{1}{7}(1,3)$	$[3,2,2]$	$\frac{1}{7}(1,5)$	$8+1/7$	$-3/7$
$\frac{1}{8}(1,1)$	$[8]$	$\frac{1}{8}(1,1)$	$8+1/4$	$-9/2$
$\frac{1}{8}(1,3)$	$[3,3]$	$\frac{1}{8}(1,3)$	$6+3/4$	-1
$\frac{1}{8}(1,5)$	$[2,3,2]$	$\frac{1}{8}(1,5)$	$8+1/4$	$-1/2$
$\frac{1}{9}(1,1)$	$[9]$	$\frac{1}{9}(1,1)$	$9+2/9$	$-49/9$
$\frac{1}{9}(1,2)$	$[5,2]$	$\frac{1}{9}(1,5)$	$7+7/9$	-2
$\frac{1}{9}(1,4)$	$[3,2,2,2]$	$\frac{1}{9}(1,7)$	$10+2/9$	$-4/9$
$\frac{1}{10}(1,1)$	$[10]$	$\frac{1}{10}(1,1)$	$10+1/5$	$-32/5$
$\frac{1}{10}(1,3)$	$[4,2,2]$	$\frac{1}{10}(1,7)$	$9+0$	$-6/5$
$\frac{1}{11}(1,1)$	$[11]$	$\frac{1}{11}(1,1)$	$11+2/11$	$-81/11$
$\frac{1}{11}(1,2)$	$[6,2]$	$\frac{1}{11}(1,6)$	$8+8/11$	$-32/11$
$\frac{1}{11}(1,3)$	$[4,3]$	$\frac{1}{11}(1,4)$	$7+7/11$	$-20/11$
$\frac{1}{11}(1,7)$	$[2,3,2,2]$	$\frac{1}{11}(1,8)$	$10+4/11$	$-6/11$
$\frac{1}{12}(1,5)$	$[3,2,3]$	$\frac{1}{12}(1,5)$	$8+5/6$	-1
$\frac{1}{12}(1,7)$	$[2,4,2]$	$\frac{1}{12}(1,7)$	$9+1/6$	$-4/3$
$\frac{1}{13}(1,2)$	$[7,2]$	$\frac{1}{13}(1,7)$	$9+9/13$	$-50/13$
$\frac{1}{13}(1,3)$	$[5,2,2]$	$\frac{1}{13}(1,9)$	$9+12/13$	$-27/13$
$\frac{1}{13}(1,4)$	$[4,2,2,2]$	$\frac{1}{13}(1,10)$	$11+1/13$	$-16/13$
$\frac{1}{13}(1,5)$	$[3,3,2]$	$\frac{1}{13}(1,8)$	$9+0$	$-15/13$

(*continued on next page*)

$\frac{1}{n}(1,q)$	$n/q=[b_1,\dots,b_s]$	$\frac{1}{n}(1,q')$	$B_{\frac{1}{n}(1,q)}$	$h_{\frac{1}{n}(1,q)}$
$\frac{1}{14}(1,3)$	[5, 3]	$\frac{1}{14}(1,5)$	$8+4/7$	$-19/7$
$\frac{1}{15}(1,2)$	[8, 2]	$\frac{1}{15}(1,8)$	$10+2/3$	$-24/5$
$\frac{1}{15}(1,4)$	[4, 4]	$\frac{1}{15}(1,4)$	$8+8/15$	$-8/3$
$\frac{1}{16}(1,3)$	[6, 2, 2]	$\frac{1}{16}(1,11)$	$10+7/8$	-3
$\frac{1}{16}(1,7)$	[3, 2, 2, 3]	$\frac{1}{16}(1,7)$	$10+7/8$	-1
$\frac{1}{16}(1,9)$	[2, 5, 2]	$\frac{1}{16}(1,9)$	$10+1/8$	$-9/4$
$\frac{1}{17}(1,2)$	[9, 2]	$\frac{1}{17}(1,9)$	$11+11/17$	$-98/17$
$\frac{1}{17}(1,3)$	[6, 3]	$\frac{1}{17}(1,6)$	$9+9/17$	$-62/17$
$\frac{1}{17}(1,4)$	[5, 2, 2, 2]	$\frac{1}{17}(1,13)$	$12+0$	$-36/17$
$\frac{1}{17}(1,5)$	[4, 2, 3]	$\frac{1}{17}(1,7)$	$9+12/17$	$-31/17$
$\frac{1}{17}(1,10)$	[2, 4, 2, 2]	$\frac{1}{17}(1,12)$	$11+5/17$	$-24/17$
$\frac{1}{18}(1,5)$	[4, 3, 2]	$\frac{1}{18}(1,11)$	$9+8/9$	-2
$\frac{1}{18}(1,7)$	[3, 3, 2, 2]	$\frac{1}{18}(1,13)$	$11+1/9$	$-11/9$
$\frac{1}{19}(1,3)$	[7, 2, 2]	$\frac{1}{19}(1,13)$	$11+16/19$	$-75/19$
$\frac{1}{19}(1,4)$	[5, 4]	$\frac{1}{19}(1,5)$	$9+9/19$	$-68/19$
$\frac{1}{19}(1,7)$	[3, 4, 2]	$\frac{1}{19}(1,11)$	$9+18/19$	$-39/19$
$\frac{1}{19}(1,8)$	[3, 2, 3, 2]	$\frac{1}{19}(1,12)$	$11+1/19$	$-22/19$
$\frac{1}{20}(1,3)$	[7, 3]	$\frac{1}{20}(1,7)$	$10+1/2$	$-23/5$
$\frac{1}{20}(1,11)$	[2, 6, 2]	$\frac{1}{20}(1,11)$	$11+1/10$	$-16/5$
$\frac{1}{21}(1,8)$	[3, 3, 3]	$\frac{1}{21}(1,8)$	$9+16/21$	$-13/7$
$\frac{1}{21}(1,13)$	[2, 3, 3, 2]	$\frac{1}{21}(1,13)$	$11+5/21$	$-4/3$
$\frac{1}{22}(1,5)$	[5, 2, 3]	$\frac{1}{22}(1,9)$	$10+7/11$	$-30/11$
$\frac{1}{23}(1,3)$	[8, 3]	$\frac{1}{23}(1,8)$	$11+11/23$	$-128/23$
$\frac{1}{23}(1,4)$	[6, 4]	$\frac{1}{23}(1,6)$	$10+10/23$	$-104/23$
$\frac{1}{23}(1,5)$	[5, 3, 2]	$\frac{1}{23}(1,14)$	$10+19/23$	$-67/23$
$\frac{1}{23}(1,7)$	[4, 2, 2, 3]	$\frac{1}{23}(1,10)$	$11+17/23$	$-42/23$
$\frac{1}{24}(1,5)$	[5, 5]	$\frac{1}{24}(1,5)$	$10+5/12$	$-9/2$
$\frac{1}{24}(1,7)$	[4, 2, 4]	$\frac{1}{24}(1,7)$	$10+7/12$	$-8/3$
$\frac{1}{25}(1,7)$	[4, 3, 2, 2]	$\frac{1}{25}(1,18)$	$12+0$	$-52/25$
$\frac{1}{25}(1,9)$	[3, 5, 2]	$\frac{1}{25}(1,14)$	$10+23/25$	-3
$\frac{1}{26}(1,7)$	[4, 4, 2]	$\frac{1}{26}(1,15)$	$10+11/13$	$-38/13$
$\frac{1}{27}(1,4)$	[7, 4]	$\frac{1}{27}(1,7)$	$11+11/27$	$-148/27$
$\frac{1}{27}(1,5)$	[6, 2, 3]	$\frac{1}{27}(1,11)$	$11+16/27$	$-11/3$
$\frac{1}{27}(1,8)$	[4, 2, 3, 2]	$\frac{1}{27}(1,17)$	$11+25/27$	-2
$\frac{1}{28}(1,5)$	[6, 3, 2]	$\frac{1}{28}(1,17)$	$11+11/14$	$-27/7$
$\frac{1}{29}(1,5)$	[6, 5]	$\frac{1}{29}(1,6)$	$11+11/29$	$-158/29$
$\frac{1}{29}(1,8)$	[4, 3, 3]	$\frac{1}{29}(1,11)$	$10+19/29$	$-79/29$
$\frac{1}{29}(1,12)$	[3, 2, 4, 2]	$\frac{1}{29}(1,17)$	$12+0$	$-60/29$
$\frac{1}{30}(1,11)$	[3, 4, 3]	$\frac{1}{30}(1,11)$	$10+11/15$	$-14/5$
$\frac{1}{31}(1,7)$	[5, 2, 4]	$\frac{1}{31}(1,9)$	$11+16/31$	$-111/31$
$\frac{1}{31}(1,11)$	[3, 6, 2]	$\frac{1}{31}(1,17)$	$11+28/31$	$-123/31$

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$\frac{1}{n}(1,q)$	$n/q=[b_1,\dots,b_s]$	$\frac{1}{n}(1,q')$	$B_{\frac{1}{n}(1,q)}$	$h_{\frac{1}{n}(1,q)}$
$\frac{1}{31}(1,12)$	$[3,3,2,3]$	$\frac{1}{31}(1,13)$	$11+25/31$	$-58/31$
$\frac{1}{33}(1,7)$	$[5,4,2]$	$\frac{1}{33}(1,19)$	$11+26/33$	$-127/33$
$\frac{1}{34}(1,9)$	$[4,5,2]$	$\frac{1}{34}(1,19)$	$11+14/17$	$-66/17$
$\frac{1}{34}(1,13)$	$[3,3,3,2]$	$\frac{1}{34}(1,21)$	$12+0$	$-35/17$
$\frac{1}{37}(1,8)$	$[5,3,3]$	$\frac{1}{37}(1,14)$	$11+22/37$	$-135/37$
$\frac{1}{39}(1,14)$	$[3,5,3]$	$\frac{1}{39}(1,14)$	$11+28/39$	$-49/13$
$\frac{1}{40}(1,11)$	$[4,3,4]$	$\frac{1}{40}(1,11)$	$11+11/20$	$-18/5$
$\frac{1}{41}(1,11)$	$[4,4,3]$	$\frac{1}{41}(1,15)$	$11+26/41$	$-151/41$

Appendix B

This appendix contains the classification of finite groups of automorphisms acting on Riemann surfaces of genus 2 and 3 so that the quotient is isomorphic to $\mathbb{P}^1$. In the last case we listed only the nonabelian groups. Tables 1–3 are adapted from [13, pages 252, 254, 255]. For every G we give a presentation, the vector $\mathbf{m}$ of branching data and the `IdSmallGroup` (G), that is the number of G in the `GAP4` database of small groups. The second author wishes to thank S.A. Broughton who kindly communicated to him that the group $G(48,33)$ (Table 3, case (3t)) was missing in [13].

Table 1
Abelian automorphism groups with rational quotient on Riemann surfaces of genus 2.

Case	G	`IdSmall Group` (G)	$\mathbf{m}$
(1a)	$\mathbb{Z}_2$	$G(2,1)$	(2^6)
(1b)	$\mathbb{Z}_3$	$G(3,1)$	(3^4)
(1c)	$\mathbb{Z}_4$	$G(4,1)$	$(2^2,4^2)$
(1d)	$\mathbb{Z}_2\times\mathbb{Z}_2$	$G(4,2)$	(2^5)
(1e)	$\mathbb{Z}_5$	$G(5,1)$	(5^3)
(1f)	$\mathbb{Z}_6$	$G(6,2)$	$(2^2,3^2)$
(1g)	$\mathbb{Z}_6$	$G(6,2)$	$(3,6^2)$
(1h)	$\mathbb{Z}_8$	$G(8,1)$	$(2,8^2)$
(1i)	$\mathbb{Z}_{10}$	$G(10,2)$	$(2,5,10)$
(1j)	$\mathbb{Z}_2\times\mathbb{Z}_6$	$G(12,5)$	$(2,6^2)$

Table 2
Nonabelian automorphism groups with rational quotient on Riemann surfaces of genus 2.

Case	G	`IdSmall Group` (G)	$\mathbf{m}$	Presentation
(2a)	$\mathcal{S}_3$	$G(6,1)$	$(2^2,3^2)$	$\langle x,y\mid x=(123),y=(12)\rangle$
(2b)	Q_8	$G(8,4)$	(4^3)	$\langle i,j,k\mid i^2=j^2=k^2=-1,\ ij=k, jk=i, ki=j\rangle$
(2c)	D_4	$G(8,3)$	$(2^3,4)$	$\langle x,y\mid x^2=y^4=1, xyx^{-1}=y^{-1}\rangle$
(2d)	$D_{4,3,-1}$	$G(12,1)$	$(3,4^2)$	$\langle x,y\mid x^4=y^3=1, xyx^{-1}=y^{-1}\rangle$
(2e)	D_6	$G(12,4)$	$(2^3,3)$	$\langle x,y\mid x^2=y^6=1, xyx^{-1}=y^{-1}\rangle$
(2f)	$D_{2,8,3}$	$G(16,8)$	$(2,4,8)$	$\langle x,y\mid x^2=y^8=1, xyx^{-1}=y^3\rangle$
(2g)	$G=\mathbb{Z}_2\ltimes((\mathbb{Z}_2)^2\times\mathbb{Z}_3)$	$G(24,8)$	$(2,4,6)$	$\langle x,y,z,w\mid x^2=y^2=z^2=w^3=1,\ [y,z]=[y,w]=[z,w]=1,\ xyx^{-1}=y, xzx^{-1}=zy, xwx^{-1}=w^{-1}\rangle$
(2h)	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24,3)$	$(3^2,4)$	$\left\langle x,y\mid x=\begin{pmatrix}1&1\\0&1\end{pmatrix}, y=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}\right\rangle$
(2i)	$\mathrm{GL}_2(\mathbb{F}_3)$	$G(48,29)$	$(2,3,8)$	$\left\langle x,y\mid x=\begin{pmatrix}1&1\\0&-1\end{pmatrix}, y=\begin{pmatrix}0&-1\\1&-1\end{pmatrix}\right\rangle$

Table 3
Nonabelian automorphism groups with rational quotient on Riemann surfaces of genus 3.

Case	G	`IdSmall Group` (G)	**m**	Presentation
(3a)	$\mathcal{S}_3$	$G(6, 1)$	$(2^4, 3)$	$\langle x, y \mid x = (12), y = (123)\rangle$
(3b)	D_4	$G(8, 3)$	$(2^2, 4^2)$	$\langle x, y \mid x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
(3c)	D_4	$G(8, 3)$	(2^5)	$\langle x, y \mid x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
(3d)	$D_{4,3,-1}$	$G(12, 1)$	$(4^2, 6)$	$\langle x, y \mid x^4 = y^3 = 1, xyx^{-1} = y^{-1}\rangle$
(3e)	D_6	$G(12, 4)$	$(2^3, 6)$	$\langle x, y \mid x^2 = y^6 = 1, xyx^{-1} = y^{-1}\rangle$
(3f)	$\mathcal{A}_4$	$G(12, 3)$	$(2^2, 3^2)$	$\langle x, y \mid x = (12)(34), y = (123)\rangle$
(3g)	$D_{2,8,5}$	$G(16, 6)$	$(2, 8^2)$	$\langle x, y \mid x^2 = y^8 = 1, xyx^{-1} = y^5\rangle$
(3h)	$D_{4,4,-1}$	$G(16, 4)$	(4^3)	$\langle x, y \mid x^4 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
(3i)	$\mathbb{Z}_2 \times D_4$	$G(16, 11)$	$(2^3, 4)$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x^2 = y^4 = 1, xyx^{-1} = y^{-1}\rangle$
(3j)	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_4)$	$G(16, 13)$	$(2^3, 4)$	$\langle x, y, z \mid x^2 = y^2 = z^4 = 1,$ $[x, z] = [y, z] = 1, xyx^{-1} = yz^2\rangle$
(3k)	$D_{3,7,2}$	$G(21, 1)$	$(3^2, 7)$	$\langle x, y \mid x^3 = y^7 = 1, xyx^{-1} = y^2\rangle$
(3l)	$D_{2,12,5}$	$G(24, 5)$	$(2, 4, 12)$	$\langle x, y \mid x^2 = y^{12} = 1, xyx^{-1} = y^5\rangle$
(3m)	$\mathbb{Z}_2 \times \mathcal{A}_4$	$G(24, 13)$	$(2, 6^2)$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x = (12)(34), y = (123)\rangle$
(3n)	$\mathrm{SL}_2(\mathbb{F}_3)$	$G(24, 3)$	$(3^2, 6)$	$\left\langle x, y \mid x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\right\rangle$
(3o)	$\mathcal{S}_4$	$G(24, 12)$	$(3, 4^2)$	$\langle x, y \mid x = (1234), y = (12)\rangle$
(3p)	$\mathcal{S}_4$	$G(24, 12)$	$(2^3, 3)$	$\langle x, y \mid x = (1234), y = (12)\rangle$
(3q)	$\mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8)$	$G(32, 9)$	$(2, 4, 8)$	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1,$ $[x, y] = [y, z] = 1, xzx^{-1} = yz^3\rangle$
(3r)	$\mathbb{Z}_2 \ltimes D_{2,8,5}$	$G(32, 11)$	$(2, 4, 8)$	$\langle x, y, z \mid x^2 = y^2 = z^8 = 1,$ $yzy^{-1} = z^5, xyx^{-1} = yz^4, xzx^{-1} = yz^3\rangle$
(3s)	$\mathbb{Z}_2 \times \mathcal{S}_4$	$G(48, 48)$	$(2, 4, 6)$	$\langle z \mid z^2 = 1\rangle \times \langle x, y \mid x = (12), y = (1234)\rangle$
(3t)	$G(48, 33)$	$G(48, 33)$	$(2, 3, 12)$	$\langle x, y, z, w, t \mid x^2 = z^2 = w^2 = t, y^3 = 1, t^2 = 1,$ $yzy^{-1} = w, ywy^{-1} = zw, zwz^{-1} = wt,$ $[x, y] = [x, z] = 1\rangle$
(3u)	$\mathbb{Z}_3 \ltimes (\mathbb{Z}_4)^2$	$G(48, 3)$	$(3^2, 4)$	$\langle x, y, z \mid x^3 = y^4 = z^4 = 1,$ $[y, z] = 1, xyx^{-1} = z, xzx^{-1} = (yz)^{-1}\rangle$
(3v)	$\mathcal{S}_3 \ltimes (\mathbb{Z}_4)^2$	$G(96, 64)$	$(2, 3, 8)$	$\langle x, y, z, w \mid x^2 = y^3 = z^4 = w^4 = 1,$ $[z, w] = 1, xyx^{-1} = y^{-1}, xzx^{-1} = w,$ $xwx^{-1} = z, yzy^{-1} = w, ywy^{-1} = (zw)^{-1}\rangle$
(3w)	$\mathrm{PSL}_2(\mathbb{F}_7)$	$G(168, 42)$	$(2, 3, 7)$	$\langle x, y \mid x = (375)(486), y = (126)(348)\rangle$

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